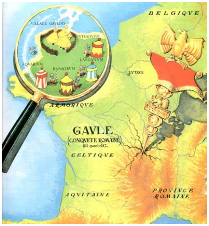
Linear Programming

Operations Research

Anthony Papavasiliou

Contents

Our Focus: $z(b)$



We care about how the optimal value of a linear program depends on the right-hand side parameters b:

$$
\begin{aligned}
\color{red}{z(b)} = \quad & \min c^T x \\
& \text{s.t. } Ax = b \\
& x \geq 0
\end{aligned}
$$

Main Takeaway of These Slides

The function $z(b)$ is a *piecewise linear* function of b

We will show this using

- the primal linear program, and
- its dual linear program

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Linear Programs in Standard Form

Linear program (LP) in standard form:

$$
\begin{aligned}
(P): \quad & \min z = c^T x \\
& \text{s.t. } Ax = b \\
& x \geq 0
\end{aligned}
$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Any LP can be expressed in standard form

Solution of an LP

Solution: a vector x such that $Ax = b$

Feasible solution: a solution with $x \geq 0$

Optimal solution: a feasible solution $x^\star$ such that $c^T x^\star \leq c^T x$ for all feasible solutions x

Basis and Basic Solution

Basis: a choice of n linearly independent columns of A

Denote $A = [B, N]$ where N are non-basic columns

Each basis corresponds to a **basic solution** $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ with $x_B = B^{-1}b$ and $x_N = 0$

Geometric property: Basic feasible solutions correspond to extreme points of the feasible region $\{x | Ax = b, x \geq 0\}$

A basis is

- **feasible** if $B^{-1}b \geq 0$
- **optimal** if feasible and $c_N^T - c_B^T B^{-1} N \geq 0$

Optimal Basis

Claim: $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ is optimal if $B^{-1}b$ and $c_N^T - c_B^T B^{-1} N \geq 0$

Proof: $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ is obviously feasible, can we improve objective function by moving away from it?

Idea: substitute basic variables for non-basic variables in objective function $c^T x$:

$$\begin{aligned} \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} &= b \Leftrightarrow \\ Bx_B + Nx_N &= b \Leftrightarrow \\ x_B &= B^{-1}(b - Nx_N) \qquad (1) \end{aligned}$$

Substituting equation (1) into the objective function,

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N \\ &= c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \end{aligned} \tag{2}$$

- Non-basic variables can only increase when moving away from the current solution while remaining feasible
- Since $c_N^T - c_B^T B^{-1} N \geq 0$, second term of equation (2) can only increase when moving in the neighborhood of the current solution $\Rightarrow$ current solution is locally optimal
- We will show later that $z(b)$ must be convex, therefore from equation (2) it must be piecewise linear

Example: The Diet Problem

Problem: Choose 3 dishes (x_1, x_2, x_3) so as to satisfy nutrient requirements b_1 and b_2, while minimizing cost (dishes cost 1 \$, 2 \$, and 1 \$ respectively)

Table: The unit of nutrients in each dish.

	Dish 1	Dish 2	Dish 3
Nutrient 1	0.5	4	1
Nutrient 2	2	1	2

$$\begin{aligned} \min \quad & x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & 0.5x_1 + 4x_2 + x_3 = b_1 \\ & 2x_1 + x_2 + 2x_3 = b_2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Example: The Diet Problem - Basic Solutions

Three possible bases:

$$B_1 = \begin{bmatrix} 0.5 & 4 \\ 2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 & 1 \\ 2 & 2 \end{bmatrix}, B_3 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Three candidate basic solutions, (parametrized on (b_1, b_2)):

$$\begin{aligned} x_{B_1} &= \begin{bmatrix} -0.1333b_1 + 0.5333b_2 \\ 0.2667b_1 - 0.0667b_2 \end{bmatrix} \\ x_{B_2} &= \begin{bmatrix} -2b_1 + b_2 \\ 2b_1 - 0.5b_2 \end{bmatrix} \\ x_{B_3} &= \begin{bmatrix} 0.2857b_1 - 0.1429b_2 \\ -0.1429b_1 + 0.5714b_2 \end{bmatrix} \end{aligned}$$

We want to understand how the objective function behaves as we change b

Example: Reduced Costs for the Diet Problem

Before starting, compute the reduced cost $c_N^T - c_B^T B^{-1} N$ for each basis:

- Basis B_1: -0.2
- Basis B_2: 1.5
- Basis B_3: 0.2143

Example: The Diet Problem - Basic Feasible Solutions

In order for a basic solution to be feasible, it is necessary that b be such that $x_B \geq 0$

Denote $R_i = \{(b_1, b_2) : x_{B_i} \geq 0\}$, then:

$$\begin{aligned} R_1 &= \{0.25b_1 \leq b_2 \leq 4b_1\} \\ R_2 &= \{2b_1 \leq b_2 \leq 4b_1\} \\ R_3 &= \{0.25b_1 \leq b_2 \leq 2b_1\} \end{aligned}$$

Example: The Diet Problem - Basic Optimal Solutions

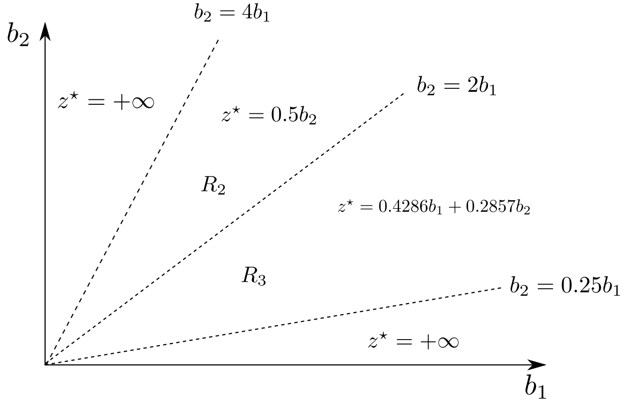
Cost of each basic solution, parametric on (b_1, b_2):

$$\begin{aligned} c_{B_1}^T x_{B_1} &= 0.4b_1 + 0.4b_2 \\ c_{B_2}^T x_{B_2} &= 0.5b_2 \\ c_{B_3}^T x_{B_3} &= 0.4286b_1 + 0.2857b_2 \end{aligned}$$

If a basis is feasible *and* has negative reduced cost, then it results in an optimal solution

From this we can infer regions over which B_2 and B_3 are optimal

Example: The Diet Problem - The Function $z(b)$



$z(b)$ is piecewise linear convex function of b

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Dual LP

The **dual** of problem (P) is the following linear program:

$$(D): \quad \max \pi^T b$$
$$\text{s.t. } \pi^T A \le c^T$$

If primal problem is not in standard form, use the following rules

Primal	Minimize	Maximize	Dual
Constraints	$\ge b_i$	≥ 0	Variables
	$\le b_i$	≤ 0	
	$= b_i$	Free	
Variables	≥ 0	$\le c_j$	Constraints
	≤ 0	$\ge c_j$	
	Free	$= c_j$	

Example: Dual of the Diet Problem

Recall the diet problem:

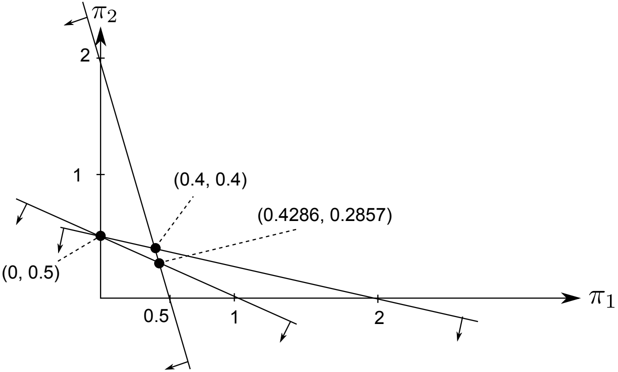
$$\begin{aligned} \min \quad & x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & 0.5x_1 + 4x_2 + x_3 = b_1 \\ & 2x_1 + x_2 + 2x_3 = b_2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Using the table in the previous slide, the dual is:

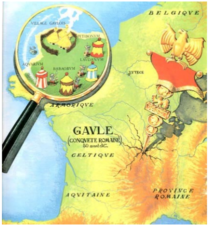
$$\begin{aligned} \max \quad & b_1\pi_1 + b_2\pi_2 \\ \text{s.t.} \quad & 0.5\pi_1 + 2\pi_2 \leq 1 \\ & 4\pi_1 + \pi_2 \leq 2 \\ & \pi_1 + 2\pi_2 \leq 1 \end{aligned}$$

Example: Dual of the Diet Problem - Feasible Region

Figure: The dual feasible region of the diet problem. Each black dot is a basic solution of the dual feasible region and corresponds to a basis of the primal problem in standard form.



Revisiting $z(b)$



Recall that we care about how the optimal value of a linear program depends on the right-hand side parameters b:

$$\begin{aligned} {\color{red}z(b)} = \quad & \min c^T x \\ & \text{s.t. } Ax = b \\ & x \geq 0 \end{aligned}$$

Rewriting the Dual Problem

Claim: $z(b)$ is a piecewise linear convex function

Proof: If a dual optimal solution exists, then one dual basic solution[1] must be optimal

Reformulation of the dual problem:

$$\max_{i=1,\ldots,r} \pi_i^T b$$

where r indexes *finitely many* basic feasible solutions

[1] General definition of **basic solution** for a polyhedron $P \subset \mathbb{R}^n$ (not necessarily in standard form) that is defined by linear equalities and inequalities: a vector x such that (i) all equality constraints are active and (ii) out of the constraints that are active at x, n are linearly independent

Computing Dual Basic Solutions

For linear programs in standard form, each basis B of the primal coefficient matrix A corresponds to a basic solution of the dual feasible set, according to the relationship

$$\pi^T = c_B^T B^{-1}$$

and vice versa

Example: Dual of the Diet Problem - Basic Solutions

Recall the three possible bases of the diet problem:

$$B_1 = \begin{bmatrix} 0.5 & 4 \\ 2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 & 1 \\ 2 & 2 \end{bmatrix}, B_3 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Basic solutions of the dual feasible region can be computed according to $\pi = c_B^T B^{-1}$:

$$\pi_1 = (0.4, 0.4), \pi_2 = (0, 0.5), \pi_3 = (0.4286, 0.2857)$$