The L-Shaped Method

Operations Research

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Extensive Form

Stochastic linear program in **extensive form**:

$$
\begin{aligned}
(EF): \quad & \min c^T x + \mathbb{E}_\omega[\min q(\omega)^T y(\omega)] \\
& Ax = b \\
& T(\omega)x + W(\omega)y(\omega) = h(\omega) \\
& x \geq 0, y(\omega) \geq 0
\end{aligned}
$$

- First-stage decisions: $x \in \mathbb{R}^{n_1}$
- second-stage decisions: $y(\omega) \in \mathbb{R}^{n_2}$
- First-stage parameters: $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage parameters: $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$, $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$ and $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

Value Function

Second-stage value function:

$$
\begin{aligned}
(S_\omega): \quad & Q_\omega(x) = \min_y q_\omega^T y \\
& W_\omega y = h_\omega - T_\omega x \\
& y \geq 0.
\end{aligned}
$$

Interpretation: cost of best possible reaction to x *and* ω

Expected value function:

$$V(x) = \sum_{\omega=1}^{N} p_\omega Q_\omega(x).$$

Interpretation: cost of best possible reaction to x *before* knowing ω

Complete Recourse

Define

- $K_1 = \{x : Ax = b, x \geq 0\}$
- $K_2(\omega) = \{x : \exists y, T_\omega x + W_\omega y = h_\omega, y \geq 0\}$
- $K_2 = \text{dom } V$

Interpretation of K_1, $K_2(\omega)$, K_2?

Relative complete recourse: obeying first-stage constraints ensures feasible second-stage decisions exist:

$$\text{pos } W = \mathbb{R}^{m_2}$$

Complete recourse: feasible second-stage decision exists, regardless of first-stage decision and realization of uncertainty:

$$K_2 = \mathbb{R}^{n_1}$$

Properties of Value Functions

Dual of (S_ω):

$$
\begin{aligned}
(D_\omega): \quad & \max_{\pi} \pi^T (h_\omega - T_\omega x) \\
& \pi^T W_\omega \leq q_\omega^T
\end{aligned}
$$

Denote $\pi_{\omega 0}$ as dual optimal multipliers of (S_ω) given x_0:

1. $V(x)$ and $Q_\omega(x)$ are piecewise linear convex functions of x
2. $\pi_{\omega 0}^T (h_\omega - T_\omega x)$ is a supporting hyperplane of $Q_\omega(x)$ at x_0
3. $\sum_{\omega=1}^{N} p_\omega \pi_{\omega 0}^T (h_\omega - T_\omega x)$ is a supporting hyperplane of $V(x)$ at x_0

We recall a previous result for the proof

Proof:

1. D_ω has finite number of dual optimal multipliers
2. Strong duality and $\pi_{\omega 0} \in \partial Q_\omega(h_\omega - T_\omega x_0)$
3. Follows from previous bullet and $V(x) = \sum_{\omega=1}^{N} p_\omega Q_\omega(x)$

The diag operator

Consider a set of matrices $A_i, i = 1, \ldots, n$ (not necessarily square)

The matrix $\text{diag}(A_1, \ldots, A_n)$ is defined as

$$\text{diag}(A_1, A_2, \ldots, A_n) = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & A_n \end{pmatrix}$$

Not necessarily a square matrix

$$(S): \quad \min_y \sum_{\omega=1}^{N} p_\omega q_\omega^T y_\omega$$
$$Wy = h - Tx$$
$$y \geq 0$$

where

- $y^T = [y_1^T, \ldots, y_N^T]$
- $h^T = [h_1^T, \ldots, h_N^T]$
- $T = \text{diag}\,(T_\omega, \omega = 1, \ldots, N)$
- $W = \text{diag}\,(W_\omega, \omega = 1, \ldots, N)$

Is there a relationship between the feasible regions of the duals of (S) and (S_ω)?

Supporting Hyperplanes of V

Denote

- V: the set of extreme vertices of $\{\pi : \pi^T W \leq q^T\}$
- V_ω: the set of extreme vertices of $\{\pi : \pi^T W_\omega \leq q_\omega^T\}$

Then

$$V = \{(p_1 \pi_1^T, \ldots, p_N \pi_N^T)^T : \pi_1 \in V_1, \ldots, \pi_N \in V_N\}$$

Domain of V

Denote

- R: the set of extreme rays of $\{\pi : \pi^T W \leq q^T\}$
- R_ω: the set of extreme rays of $\{\pi : \pi^T W_\omega \leq q_\omega^T\}$

Then

$$R = \{(0, \ldots, \sigma_\omega^T, \ldots, 0)^T : \sigma_\omega \in R_\omega, \omega = 1, \ldots, N\}$$

Deterministic Equivalent Program

The original problem (EF) can be written as a **deterministic equivalent program**:

$$
\begin{aligned}
&\min c^T x + \theta \\
&Ax = b \\
&\sigma^T (h - Tx) \leq 0, \sigma \in R \\
&\theta \geq \pi^T (h - Tx), \pi \in V \\
&x \geq 0
\end{aligned}
$$

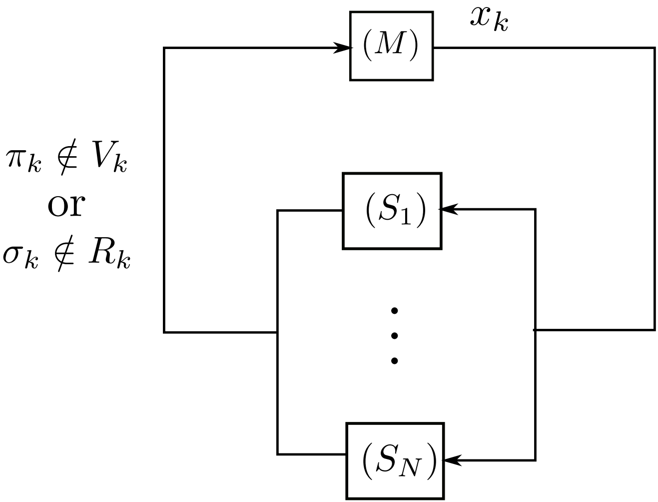
Master Problem

Define **master problem** as

$$
\begin{aligned}
(M): \quad & z_k = \min c^T x + \theta \\
& Ax = b \\
& \sigma^T (h - Tx) \leq 0, \sigma \in R_k \subseteq R \qquad (1) \\
& \theta \geq \pi^T (h - Tx), \pi \in V_k \subseteq V \qquad (2) \\
& x \geq 0
\end{aligned}
$$

- Feasibility cuts: equation 1
- Optimality cuts: equation 2

Overall Scheme



Bounds

Solution of master provides:

- lower bound $z_k \leq z^\star$
- candidate solution x_k
- under-estimator $\theta_k \leq V(x_k)$

Solution of *all* (S_ω) with input x_k provides

- upper bound $c^T x_k + \sum_{\omega=1}^{N} p_\omega q_\omega^T y_{\omega,k+1} \geq z^\star$
- new vertex $\pi_{k+1} = (p_1 \pi_{1,k+1}^T, \dots, p_N \pi_{N,k+1}^T)^T$ or new extreme ray $\sigma_{k+1} = (0, \dots, \sigma_\omega^T, \dots, 0)^T$

The L-Shaped Algorithm

Step 0: Set $k = 0,\ V_0 = R_0 = \emptyset$

Step 1: Solve (M)

- If (M) is feasible, store x_k
- If (M) is infeasible, exit: infeasible

Step 2: For $\omega = 1, \ldots, N$, solve (S_ω) with x_k as input

- If (S_ω) is infeasible, let $S_{k+1} = S_k \cup \{\sigma_{k+1}\}$, where σ_{k+1} is an extreme ray of (S_ω), let $k = k + 1$ and return to step 1
- If (S_ω) is feasible, store $\pi_{\omega,k+1}$

Step 3: Let $V_{k+1} = V_k \cup \{(p_1\pi_{1,k+1}, \ldots, p_N\pi_{N,k+1})\}$

- If $V_k = V_{k+1}$ then terminate with (x_k, y_{k+1}) as the optimal solution.
- Else, let $k = k + 1$ and return to step 1

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Example: Capacity Expansion Planning

$$\min_{x,y\geq 0} \sum_{i=1}^{n} (I_i \cdot x_i + \mathbb{E}_\xi \sum_{j=1}^{m} C_i \cdot T_j \cdot y_{ij}(\omega))$$

$$\text{s.t.} \sum_{i=1}^{n} y_{ij}(\omega) = D_j(\omega), j = 1, \ldots, m$$

$$\sum_{j=1}^{m} y_{ij}(\omega) \leq x_i, i = 1, \ldots n - 1$$

- I_i, C_i: fixed/variable cost of technology i
- $D_j(\omega)$, T_j: height/width of load block j
- $y_{ij}(\omega)$: capacity of i allocated to j
- x_i: capacity of i

Note: D_j is not dependent on ω

Problem Data

Two possible realizations of load duration curve:

- Reference scenario: 10%
- 10x wind scenario: 90%

	Duration (hours)	Level (MW) Reference scenario	Level (MW) 10x wind scenario
Base load	8760	0-7086	0-3919
Medium load	7000	7086-9004	3919-7329
Peak load	1500	9004-11169	7329-10315

Slave Problem

$$(S_\omega): \quad \min_{y \geq 0} \sum_{i=1}^{n} \sum_{j=1}^{m} C_i \cdot T_j \cdot y_{ij}$$

$$(\lambda_j(\omega)): \quad \sum_{i=1}^{n} y_{ij} = D_j(\omega), j = 1, \ldots, m$$

$$(\rho_i(\omega)): \quad \sum_{j=1}^{m} y_{ij} \leq \bar{x}_i, i = 1, \ldots n-1$$

where $\bar{x}$ has been fixed from the master problem

Sequence of Investment Decisions

Iteration	Coal (MW)	Gas (MW)	Nuclear (MW)	Oil (MW)
1	0	0	0	0
2	0	0	0	8736
3	0	0	0	15999.6
4	0	14675.5	0	0
5	10673.8	0	0	0
6	10673.8	0	0	13331.8
7	0	163.8	7174.5	3830.8
8	0	3300.6	7868.4	0
9	0	5143.4	7303.9	1679.4
10	3123.9	1948.1	4953.7	1143.3
11	1680	4322.4	6625	0
12	8747.6	1652.8	0	768.6
13	5701.9	464.9	4233.6	768.6
14	4935.9	1405	3994.7	0
15	6552.6	386.3	3173.7	882.9
16	5085	1311	3919	854

Observations

- Investment candidate in each iteration necessarily different from *all* past iterations
- ’Greedy’ behavior: low capital cost in early iterations

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Example 1

$$z = \min 100x_1 + 150x_2 + \mathbb{E}_\xi(q_1 y_1 + q_2 y_2)$$

$$\text{s.t. } x_1 + x_2 \leq 120$$

$$6y_1 + 10y_2 \leq 60x_1$$

$$8y_1 + 5y_2 \leq 80x_2$$

$$y_1 \leq d_1, y_2 \leq d_2$$

$$x_1 \geq 40, x_2 \geq 20, y_1, y_2 \geq 0$$

$$\xi = (d_1, d_2, q_1, q_2) = \begin{cases} (500, 100, -24, -28), & p_1 = 0.4 \\ (300, 300, -28, -32), & p_2 = 0.6 \end{cases}$$

Iteration 1

- *Step 1.*
  $\min\{100x_1 + 150x_2 | x_1 + x_2 \leq 120, x_1 \geq 40, x_2 \geq 20\}$
- $x^1 = (40, 20)^T$, $\theta^1 = -\infty$
- *Step 3.* For $\xi = \xi_1$ solve

$$\min\{-24y_1 - 28y_2 | 6y_1 + 10y_2 \leq 2400, 8y_1 + 5y_2 \leq 1600$$
$$0 \leq y_1 \leq 500, 0 \leq y_2 \leq 100\}$$
$$w_1 = -6100, y^T = (137.5, 100), \pi_1^T = (0, -3, 0, -13)$$

For $\xi = \xi_2$ solve

$$\min\{-28y_1 - 32y_2 | 6y_1 + 10y_2 \leq 2400, 8y_1 + 5y_2 \leq 1600$$
$$0 \leq y_1 \leq 300, 0 \leq y_2 \leq 300\}$$
$$w_2 = -8384, y^T = (80, 192), \pi_2^T = (-2.32, -1.76, 0, 0)$$

Iteration 1: Optimality Cut

$h_1 = (0, 0, 500, 100)^T$, $h_2 = (0, 0, 300, 300)^T$
$T_{\cdot,1} = (-60, 0, 0, 0)^T$, $T_{\cdot,2} = (0, -80, 0, 0)^T$

- $e_1 = 0.4 \cdot \pi_1^T \cdot h_1 + 0.6 \cdot \pi_2^T \cdot h_2 = 0.4 \cdot (-1300) + 0.6 \cdot (0) = -520$
- $E_1 = 0.4 \cdot \pi_1^T T + 0.6 \cdot \pi_2^T T =$ $0.4(0, 240) + 0.6(139.2, 140.8) = (83.52, 180.48)$
- $w^1 = -520 - (83.52, 180.48) \cdot x^1 = -7470.4$
- $w^1 = -7470.4 > \theta^1 = -\infty$, therefore add the cut $83.52x_1 + 180.48x_2 + \theta \geq -520$

Iteration 2

- *Step 1.* Solve master

$$\min\{100x_1 + 150x_2 + \theta | x_1 + x_2 \leq 120, x_1 \geq 40, x_2 \geq 20,$$
$$83.52x_1 + 180.48x_2 + \theta \geq -520\}$$
$$z = -2299.2, x^2 = (40, 80)^T, \theta^2 = -18299.2$$

- *Step 3.* Add the cut $211.2x_1 + \theta \geq -1584$

Iteration 3

- *Step 1.* Solve master.

$$z = -1039.375, x^3 = (66.828, 53.172)^T, \theta^3 = -15697.994$$

- *Step 3.* Add the cut $115.2x_1 + 96x_2 + \theta \geq -2104$

Iteration 4

- *Step 1.* Solve master.

$$z = -889.5, x^4 = (40, 33.75)^T, \theta^4 = -9952$$

- *Step 3.* There are multiple solutions for $\xi = \xi_2$. Select one, add the cut $133.44x_1 + 130.56x_2 + \theta \geq 0$

Iteration 5

- *Step 1.* Solve master

$$\min\{100x_1 + 150x_2 + \theta | x_1 + x_2 \leq 120, x_1 \geq 40, x_2 \geq 20,$$
$$83.52x_1 + 180.48x_2 + \theta \geq -520, 211.2x_1 + \theta \geq -1584$$
$$115.2x_1 + 96x_2 + \theta \geq -2104, 133.44x_1 + 130.56x_2 + \theta \geq 0\}$$
$$z = -855.833, x^5 = (46.667, 36.25)^T, \theta^5 = -10960$$

- *Step 3.* $w_5 = -520 - (83.52, 180.48) \cdot x^5 = -10960 = \theta^5$, stop. $x = (46.667, 36.25)^T$ is the optimal solution.

Example 2

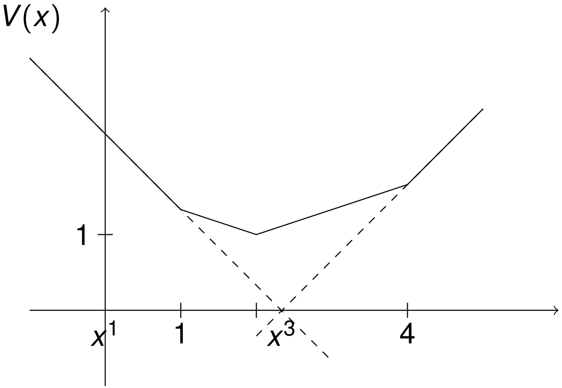
$$z = \min \mathbb{E}_\xi(y_1 + y_2)$$
$$\text{s.t. } 0 \le x \le 10$$
$$y_1 - y_2 = \xi - x$$
$$y_1, y_2 \ge 0$$

$$\xi = \begin{cases} 1 & p_1 = 1/3 \\ 2 & p_2 = 1/3 \\ 4 & p_3 = 1/3 \end{cases}$$

$K_2 = \mathbb{R}$

L-Shaped Method in Example 2

- Iteration 1, Step 1: $x^1 = 0$
- Iteration 1, Step 3: x^1 not optimal, add cut: $\theta \geq 7/3 - x$
- Iteration 2, Step 1: $x^2 = 10$
- Iteration 2, Step 3: x^2 not optimal, add cut: $\theta \geq x - 7/3$
- Iteration 3, Step 1: $x^3 = 7/3$
- Iteration 3, Step 3: x^3 not optimal, add cut: $\theta \geq (x + 1)/3$
- Iteration 4, Step 1: $x^4 = 1.5$
- Iteration 4, Step 3: x^4 not optimal, add cut: $\theta \geq (5 - x)/3$
- Iteration 3, Step 1: $x^5 = 2$
- Iteration 3, Step 3: x^5 is optimal



- $V(x^1) = 7/3$ and $V(x) = 7/3 - x$ 'around' x^1
- $(7 - x)/3$ is the optimality cut at x^1

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Feasibility Cuts

Consider the following problem:

$$
\begin{aligned}
&(F): \min w' = e^T v^+ + e^T v^- \\
&\text{s.t. } Wy + Iv^+ - Iv^- = h_k - T_k x^v \\
&y \geq 0, v^+ \geq 0, v^- \geq 0
\end{aligned}
$$

with dual multipliers σ^v. Define

$$
\begin{aligned}
D_{r+1} &= (\sigma^v)^T T_k \\
d_{r+1} &= (\sigma^v)^T h_k
\end{aligned}
$$

Step 2 of L-shaped method: For $k = 1, \ldots, K$ solve (F).

- If $w' = 0$ for all k, go to Step 3.
- Else, add $D_{r+1} x \geq d_{r+1}$, set $r = r + 1$ and go to Step 1.

Example

$$\min 3x_1 + 2x_2 - \mathbb{E}_\xi(15y_1 + 12y_2)$$
$$\text{s.t. } 3y_1 + 2y_2 \le x_1, 2y_1 + 5y_2 \le x_2$$
$$0.8\xi_1 \le y_1 \le \xi_1, 0.8\xi_2 \le y_2 \le \xi_2$$
$$x, y \ge 0$$

$$\xi = \begin{cases} (4,4), p_1 = 0.25 \\ (4,8), p_2 = 0.25 \\ (6,4), p_3 = 0.25 \\ (6,8), p_4 = 0.25 \end{cases}$$

Generating a Feasibility Cut

For $x^1 = (0, 0)^T$, $\xi = (6, 8)^T$, solve

$$\min_{v^+, v^-, y} v_1^+ + v_1^- + v_2^+ + v_2^- + v_3^+ + v_3^- +$$
$$v_4^+ + v_4^- + v_5^+ + v_5^- + v_6^+ + v_6^-$$
$$\text{s.t. } v_1^+ - v_1^- + 3y_1 + 2y_2 \leq 0, v_2^+ - v_2^- + 2y_1 + 5y_2 \leq 0$$
$$v_3^+ - v_3^- + y_1 \geq 4.8, v_4^+ - v_4^- + y_2 \geq 6.4$$
$$v_5^+ - v_5^- + y_1 \leq 6, v_6^+ - v_6^- + y_2 \leq 8$$

We get $w' = 11.2$, $\sigma^1 = (-3/11, -1/11, 1, 1, 0, 0)$
$h = (0, 0, 4.8, 6.4, 6, 8)^T$, $T_{\cdot,1} = (-1, 0, 0, 0, 0, 0)^T$,
$T_{\cdot,2} = (0, -1, 0, 0, 0, 0)^T$
$D_1 = (-3/11, -1/11, 1, 1, 0, 0) \cdot T = (3/11, 1/11)$,
$d_1 = (-3/11, -1/11, 1, 1, 0, 0) \cdot h = 11.2$
$3/11 x_1 + 1/11 x_2 \geq 11.2$

Induced Constraints

Going by the book:

- Iteration 2 master problem: $x^2 = (41.067, 0)^T$
- Iteration 2 feasibility cut: $x_2 \geq 22.4$
- Iteration 3 master problem: $x^3 = (33.6, 22.4)^T$
- Iteration 3 feasibility cut: $x_2 \geq 41.6$
- Iteration 4 master problem: $x^4 = (27.2, 41.6)^T$ is feasible

Induced Constraints:

- Observe that for $\xi = (6, 8)^T$, $y_1 \geq 4.8$, $y_2 \geq 6.4$
- This implies $x_1 \geq 27.2$, $x_2 \geq 41.6$, which should be added directly to the master

Personal experience: feasibility cuts are impractical