The Power Flow Problem

Quantitative Energy Economics

Anthony Papavasiliou

The Power Flow Problem

Table of Contents

Incidence Matrix

Consider a graph $G = (N, E)$, where N is the set of nodes and E is the set of edges

The **incidence matrix** of the graph is defined as $A = (A_{ij}), i, j \in N$, where

$$A_{ij} = \begin{cases} 1, & \text{if } (i, j) \in E, \\ 0, & \text{otherwise} \end{cases}$$

Degree Matrix

Consider a graph $G = (N, E)$

The **degree matrix** is defined as $D = (D_{ij}), i, j \in N$, where

$$D_{ij} = \begin{cases} d_i, & \text{if } (i = j), \\ 0, & \text{otherwise} \end{cases}$$

d_i is the degree of node i, which is the number of edges that are incident to the node

For weighted graphs the degree generalizes to the sum of the weights of the edges that are indecent to the node

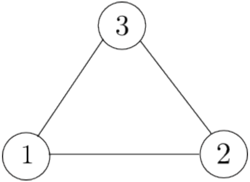
Graph Laplacian

Consider a graph $G = (N, E)$, where A is its incidence matrix and D is its degree matrix

The **Laplacian** of the graph is then defined as

$$L = D - A$$

Example: Three-Node Graph



Incidence matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Example: Three-Node Graph

Degree matrix:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Laplacian:

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Laplacian Matrix is Positive Semi-Definite

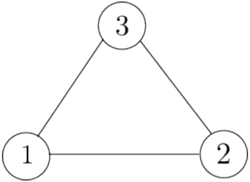
A matrix $A \in \mathbb{R}^{n \times n}$ is **positive semidefinite** if, for any non-zero vector $f \in \mathbb{R}^n$, $f^T A f \geq 0$

Consider a graph $G = (N, E)$, and pick any vector $f \in \mathbb{R}^{|N|}$

The i-th component of Lf is $(Lf)_i = \sum_{j \in N:(i,j) \in E} (f_i - f_j)$

$$\begin{aligned}
f^T L f &= \sum_{i \in N} f_i \sum_{j \in N:(i,j) \in E} (f_i - f_j) \\
&= \sum_{(i,j) \in E} f_i (f_i - f_j) + \sum_{(i,j) \in E} f_j (f_j - f_i) \\
&= \sum_{(i,j) \in E} (f_i (f_i - f_j) - f_j (f_i - f_j)) \\
&= \sum_{(i,j) \in E} (f_i - f_j)^2 \geq 0
\end{aligned}$$

Example: Three-Node Graph



Consider any non-zero vector $f = (f_1, f_2, f_3)$:

$$Lf = \begin{bmatrix} 2f_1 - f_2 - f_3 \\ 2f_2 - f_1 - f_3 \\ 2f_3 - f_1 - f_2 \end{bmatrix} = \begin{bmatrix} (f_1 - f_2) + (f_1 - f_3) \\ (f_2 - f_1) + (f_2 - f_3) \\ (f_3 - f_1) + (f_3 - f_2) \end{bmatrix}.$$

Example: Three-Node Graph

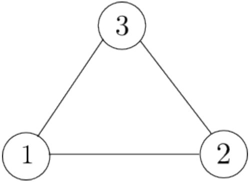
Left-multiplying by f,

$$\begin{aligned} f^T L f &= f_1(f_1 - f_2) + f_1(f_1 - f_3) \\ &\quad + f_2(f_2 - f_1) + f_2(f_2 - f_3) \\ &\quad + f_3(f_3 - f_1) + f_3(f_3 - f_2) \\ &= (f_1 - f_2)^2 + (f_2 - f_3)^2 + (f_1 - f_3)^2 \end{aligned}$$

Graph Laplacian Theorem

The multiplicity of the eigenvalue $\lambda = 0$ in the Laplacian of a graph is equal to the number of connected components of the graph

Example: Three-Node Graph



Laplacian:

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Eigenvalues of L: $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 3$

Since the graph is connected, it has a single eigenvalue equal to zero

Table of Contents

Circuits

Electric circuits consist of

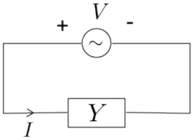
- **passive elements** that act as carriers of electric power (transmission lines, transformers)
- **active elements** that generate or consume electric power (generators, loads)

State of a circuit can be described by:

- current along every branch
- voltage difference between each node of the circuit and a reference point called **ground**

Once we know the state of the system, we know *everything* about the circuit

Ohm's Law



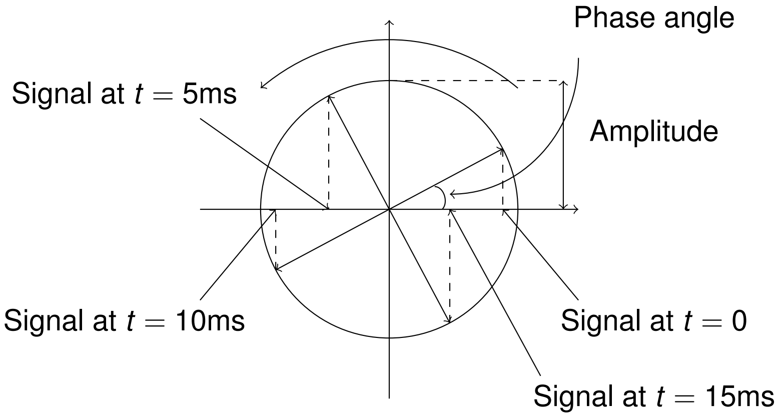
$$I = YV, S = VI^{\star} = P + Qi$$

Passive electric elements are characterized by an **admittance**, Y, which is determined by their electrical characteristics

Denote V as the voltage applied along the terminals of a passive element, and I as the current flowing across it. **Ohm's law** requires that:

$$I = Y \cdot V$$

Complex Representation of a Sinusoidal Signal



Alternating Current (AC) Electric Power Systems

AC power systems have voltage and current that fluctuate sinusoidally (50 Hz in Europe, 60 Hz in USA)

Sinusoidal signals can be described as complex number:

- **Amplitude** of the signal: magnitude of the vector
- **Phase angle**: angle of the complex number with respect to the horizontal axis

Example: consider a current signal $I = (3 + 4 \cdot i)$ Ampere

- Magnitude: 5 Ampere
- Phase angle: 36.9 degrees

Example: Voltage and Current of a Passive Element

Consider applying a voltage of 230 V across a passive element with $Y = 0.01 - 0.01 \cdot i$

By Ohm's law:

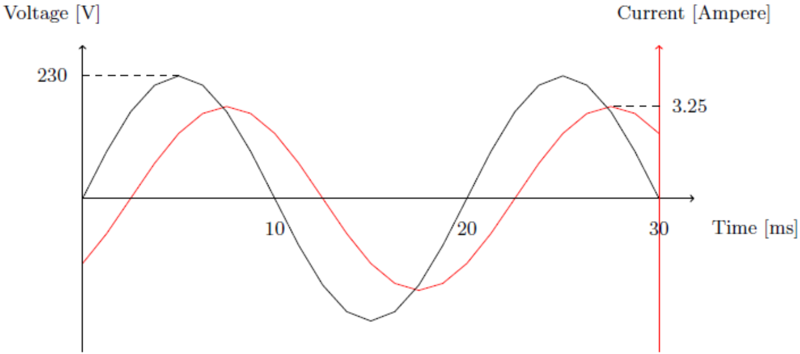
$$I = V \cdot Y = 230 \cdot (0.01 - 0.01 \cdot i) = 2.3 - 2.3 \cdot i$$

Amplitude of current: $\sqrt{2.3^2 + 2.3^2} = 3.25$ Ampere

Phase difference between current and voltage: $\arctan(\frac{-2.3}{2.3}) = -45^\circ$

Current lags voltage by $\frac{45}{360}$ of a full cycle (20ms), hence current peaks 2.5 ms after voltage

Example: Voltage and Current of a Passive Element



Power Flow

Consider a branch (m, n) of a circuit, V_{mn} the voltage applied across the branch, I_{mn} the current flowing across an element

We can define the (i) **apparent power** S_{mn}, (ii) **real power** P_{mn}, and (iii) **reactive power** Q_{mn} consumed on the line as follows:

$$S_{mn} = P_{mn} + Q_{mn} \cdot i = V_{mn} \cdot I^{\star}_{mn}$$

Resistors, Inductors, Capacitors

Classification of passive electrical equipment based on admittance $Y = G + Bi$, where G is the **conductance**, and B is the **susceptance**:

- Resistors: positive conductance ($G > 0, B = 0$), consume real power ($P_{mn} > 0$)
- Inductors: negative susceptance ($B < 0, G = 0$), consume reactive power ($Q_{mn} > 0$)
- Capacitors: positive susceptance ($G = 0, B > 0$), produce reactive power

Typically, transmission lines and transformers are reactive (i.e. $B < 0$) and slightly resistive (i.e. $G > 0$ but $G >> B$)

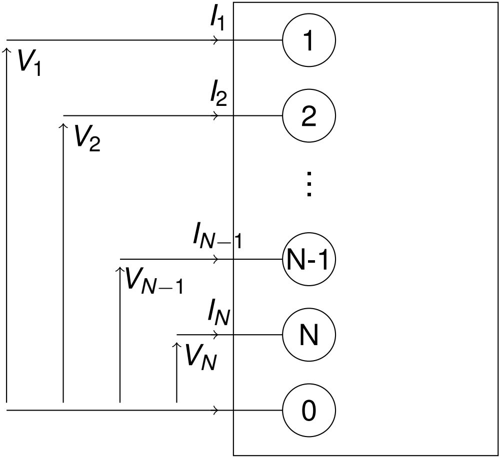
Table of Contents

Kirchoff's Laws

Kirchoff's current law: total amount of current flowing into a node equals the total amount of current flowing out of a node

Kirchoff's voltage law: accumulated voltage change across any loop of an electrical circuit equals zero

N-Port Model of Circuit



$$S_m = V_m \cdot I_m^\star = P_m + Q_m \cdot i$$

Performance Equations in Admittance Form

Denote bus currents as $I_{\text{bus}} = (I_1, I_i, \ldots, I_N)^T$, bus voltages as $V_{\text{bus}} = (V_1, V_2, \ldots, V_N)^T$ and Y_{bus}

Define the **admittance matrix** of the network:

- Y_{mn} is the negative of the admittance between bus m and bus n for $m \neq n$
- Y_{mm} is the sum of the admittance between node m and the ground plus the admittance between node m and all of its adjacent nodes for $m = n$

Performance equations in admittance form:

$$I_{\text{bus}} = Y_{\text{bus}} \cdot V_{\text{bus}}$$

Power Flow Equations

Kirchhoff's current law at the *m*-th bus:

$$I_m = \sum_{n=1}^{N} Y_{mn} \cdot V_n$$

Conjugating this equation, we get

$$S_m = V_m \cdot I_m^\star = V_m \cdot \sum_{n=1}^{N} Y_{mn}^\star \cdot V_n^\star$$

Separating into real and imaginary parts, we get **power flow equations**

$$\begin{aligned} P_m &= Re(V_m \cdot I_m^\star) = Re(V_m \cdot \sum_{n=1}^{N} Y_{mn}^\star \cdot V_n^\star), \\ Q_m &= Im(V_m \cdot I_m^\star) = Im(V_m \cdot \sum_{n=1}^{N} Y_{mn}^\star \cdot V_n^\star) \end{aligned}$$

Power Flow Equations in Polar Coordinates

In polar coordinates,

$$
\begin{aligned}
P_m &= |V_m| \cdot \sum_{n=1}^{N} |V_n| \cdot (G_{mn} \cdot \cos(\theta_{mn}) + B_{mn} \cdot \sin(\theta_{mn})), \\
Q_m &= |V_m| \cdot \sum_{n=1}^{N} |V_n| \cdot (G_{mn} \cdot \sin(\theta_{mn}) - B_{mn} \cdot \cos(\theta_{mn})),
\end{aligned}
$$

where $Y_{mn} = G_{mn} + B_{mn} \cdot i$ and θ_{mn} is the phase angle difference of voltages V_m and V_n

Linearized Power Flow Equations

Consider the following approximations:

- Line resistance is negligible: $G_{mn} = 0$
- Phase angles across branches $\theta_{mn} = \theta_m - \theta_n$ are sufficiently small: $\sin(\theta_{mn}) \simeq \theta_{mn}$ and $\cos(\theta_{mn}) \simeq 1$
- Voltage magnitude on each bus is nominal: $|V_m| \simeq 1$

This results in the **linearized power flow equations**

Ignoring the Ground Node

- When a network has no passive element connected to the ground node, the graph of the system can be simplified by ignoring the ground node
- In this case, an $N+1$-node network is referred to as an N-node network
- Recall that a generator cannot be injecting power into the ground node

Example

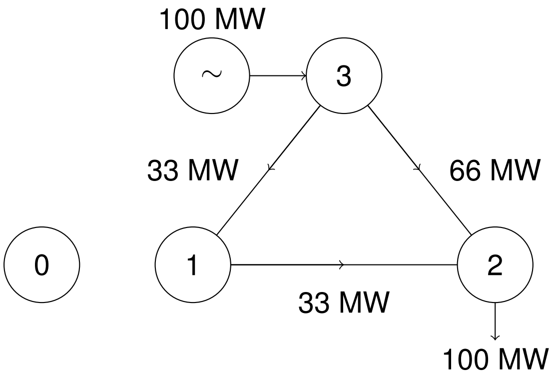


Figure: A 4-node network with no passive element connected to ground can be represented as a 3-node network by ignoring the ground node.

Table of Contents

Linearized Power Flow Equations

Linearization of power flow equations implies

$$P_m = \sum_{n=1}^{N} B_{mn}(\theta_m - \theta_n)$$

Rewrite in terms of **reactance** $X_{mn} = B_{mn}^{-1}$:

$$P_m = \sum_{n=1, n \neq m}^{N} \frac{1}{X_{mn}} \theta_m - \sum_{n=1, n \neq m}^{N} \frac{1}{X_{mn}} \theta_n$$

Node-Based Direct Current Power Flow Equations

Denote $T = (T_{mn}), m, n \in \{1, \ldots, N\} - \{h\}$, where h is the hub node and T is an $N-1 \times N-1$ matrix whose elements are

$$T_{mn} = \begin{cases} -\frac{1}{X_{mn}} & (m,n) \in A, m \neq n \\ \displaystyle\sum_{n'=1, n' \neq m}^{N} \frac{1}{X_{mn'}} & m = n \\ 0 & (m,n) \notin A \end{cases}$$

where A is the set of arcs in the network

Node-Based Direct Current Power Flow Equations

Compactly:

$$P = T\theta$$

where $P = (P_m), m \in \{1, \dots, N\} - \{h\}$ and
$\theta = (\theta_m), m \in \{1, \dots, N\} - \{h\}$

Adding conservation of energy,

$$P_h = - \sum_{n \in \{1, \dots, N\} - \{h\}} P_n$$

we have the **node-based direct current power flow equations**

Observations

- *Important observation*: Dependence of power injections, P_m, to bus angles, θ_m, described by a matrix which is the weighted Laplacian of the graph of the electric network, where the weights on the lines are given by B_{mn}
- From graph Laplacian theorem, if the graph is connected, then the Laplacian has rank $N-1$
- Conclusion: fixing the phase angle of the hub node, power injections uniquely determine the remaining phase angles
- *Node-based* equations because power flows are expressed as a function of *nodal* phase angle differences
- *Lossless* DC power flow model: neglects thermal losses on lines, since it assumes that the resistance of the passive elements is zero

Table of Contents

Line Flow as a Function of Bus Angles

Proposition: Power flow on line (m, n) is given as

$$P_{mn} = \frac{\theta_m - \theta_n}{X_{mn}}$$

Power Transfer Distribution Factors

Given a line k and a bus n, the **power transfer distribution factor (PTDF)** is the amount of power flow induced on line k by a transfer of power from bus n to the hub node

Recall: the value of a PTDF is therefore dependent on the choice of hub node

Computing PTDFs

Define the matrix M as $M = (M_{kn}), k \in E, n \in N - \{h\}$ where

$$M_{kn} = \begin{cases} \frac{1}{X_k}, & \text{if } k = (n, \cdot), n \neq h, \\ -\frac{1}{X_k}, & \text{if } k = (\cdot, n), n \neq h, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of M, and from the previous proposition

$$P_L = M\theta,$$

where P_L is the vector of power flows along the lines of the network

Computing PTDFs

From the node-based power flow equations,

$$P_L = MT^{-1}P,$$

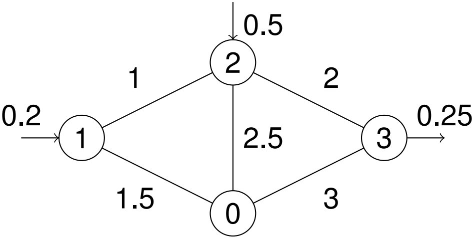
where $P = (P_n), n \in \{1, \ldots, N\} - \{h\}$ is an injection of power from bus n to the hub node

The PTDF of bus n on line k (denoted as F_{kn}) is obtained as:

$$F_{kn} = M'_k(T^{-1})_n,$$

where M_k is the k-th row of M, and $(T^{-1})_n$ is the n-th column of the matrix T^{-1}

Example: 4-Bus Network



Example: 4-Bus Network

Problem: Compute the PTDF from bus 1 to line 2-3 when 0 is the hub node

Solution:

Start by computing T matrix:

$$T = \begin{bmatrix} \frac{1}{1} + \frac{1}{1.5} & -\frac{1}{1} & 0 \\ -\frac{1}{1} & \frac{1}{1} + \frac{1}{2.5} + \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} + \frac{1}{3} \end{bmatrix}$$

Invert T:

$$T^{-1} = \begin{bmatrix} 0.96 & 0.6 & 0.36 \\ 0.6 & 1.0 & 0.6 \\ 0.36 & 0.6 & 1.56 \end{bmatrix}$$

For power injection $P = (0.2, 0.5, -0.25)^T$, bus angles are

$$\theta = T^{-1}P = \begin{bmatrix} 0.96 & 0.6 & 0.36 \\ 0.6 & 1.0 & 0.6 \\ 0.36 & 0.6 & 1.56 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.5 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 0.402 \\ 0.470 \\ -0.018 \end{bmatrix}$$

The power flow on each line is

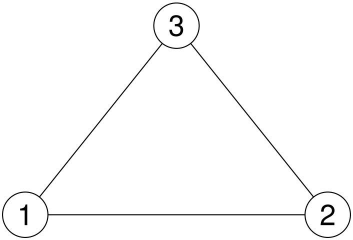
$$\begin{aligned} P_{12} &= \frac{\theta_1 - \theta_2}{X_{12}} = \frac{0.402 - 0.47}{1} = -0.068, \\ P_{10} &= \frac{\theta_1 - \theta_0}{X_{10}} = \frac{0.402 - 0}{1.5} = 0.268, \\ P_{23} &= \frac{\theta_2 - \theta_3}{X_{23}} = \frac{0.47 - (-0.018)}{2} = 0.244, \\ P_{20} &= \frac{\theta_2 - \theta_0}{X_{20}} = \frac{0.47 - 0}{2.5} = 0.188, \\ P_{30} &= \frac{\theta_3 - \theta_0}{X_{30}} = \frac{-0.018 - 0}{3} = -0.006. \end{aligned}$$

The matrix M that determines line flows as a function of bus angles, $P_L = M\theta$, is

$$M = \left[\begin{array}{c|ccc} 0-1 & -\frac{1}{1.5} & 0 & 0 \\ 1-2 & \frac{1}{1} & -\frac{1}{1} & 0 \\ 2-3 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0-2 & 0 & -\frac{1}{2.5} & 0 \\ 0-3 & 0 & 0 & -\frac{1}{3} \end{array}\right]$$

PTDF of line $(2, 3)$ for bus 1 is

$$F_{2-3,1} = M_{2-3}(T^{-1})_1 = (0, 0.5, -0.5) \cdot (0.96, 0.6, 0.36)^T = 0.12$$



Denote X as the reactance of each line

Problem: Compute the PTDF matrix of the network when node 3 is the hub node.

Example: Symmetric 3-Node Network

Compute matrix T:

$$T = \begin{bmatrix} \frac{1}{X} + \frac{1}{X} & -\frac{1}{X} \\ -\frac{1}{X} & \frac{1}{X} + \frac{1}{X} \end{bmatrix} = \frac{1}{X} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Invert matrix T:

$$T^{-1} = X \begin{bmatrix} 0.667 & 0.333 \\ 0.333 & 0.667 \end{bmatrix}$$

Compute matrix M:

$$M = \left[\begin{array}{c|cc} 1-2 & \frac{1}{X} & -\frac{1}{X} \\ 2-3 & 0 & \frac{1}{X} \\ 1-3 & \frac{1}{X} & 0 \end{array}\right]$$

Compute PTDFs:

$$
\begin{aligned}
F_{1-2,1} &= (\frac{1}{X}, -\frac{1}{X}) \cdot (0.667X, 0.333X)^T = 0.333, \\
F_{1-3,1} &= (\frac{1}{X}, 0) \cdot (0.667X, 0.333X)^T = 0.667, \\
F_{2-3,1} &= (0, \frac{1}{X}) \cdot (0.667X, 0.333X)^T = 0.333, \\
F_{1-2,2} &= (\frac{1}{X}, -\frac{1}{X}) \cdot (0.333X, 0.667X)^T = -0.333, \\
F_{1-3,2} &= (\frac{1}{X}, 0) \cdot (0.333X, 0.667X)^T = 0.333, \\
F_{2-3,2} &= (0, \frac{1}{X}) \cdot (0.333X, 0.667X)^T = 0.667.
\end{aligned}
$$

Physical intuition: current splits in a way which is inversely proportional to reactance