Stochastic Linear Programming

Operations Research

Anthony Papavasiliou

Stochastic Linear Programming

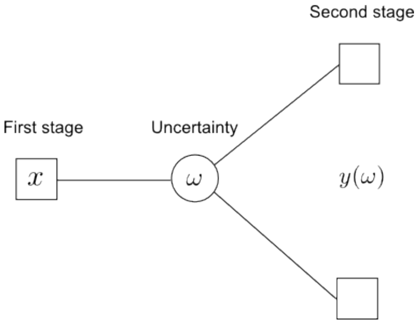
Outline

2-Stage Stochastic Linear Programs

- **Stochastic linear programs**: linear programs with uncertain data
- Key feature of stochastic programs is **recourse**: decisions that can be taken after uncertainty is revealed

Sequence of Events

1. **First-stage decisions**: decisions taken before uncertainty is revealed
2. **Second-stage decisions**: decisions taken after uncertainty is revealed
3. Sequence of events: $x \to \omega \to y(x, \omega)$



Mathematical Formulation

$$\min c^T x + \mathbb{E}_\omega[\min q(\omega)^T y(\omega)]$$
$$Ax = b$$
$$T(\omega)x + W(\omega)y(\omega) = h(\omega)$$
$$x \geq 0, y(\omega) \geq 0$$

- First stage decisions $x \in \mathbb{R}^{n_1}$, second stage decisions $y(\omega) \in \mathbb{R}^{n_2}$
- First stage parameters: $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage data: $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$, $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$, $W(\omega) \in \mathbb{R}^{m_2,n_2}$
- **Fixed recourse** if W does not depend on ω

Example: Newsboy Problem

Denote

- x: amount of product produced in period 1
- y: amount of product sold in period 2
- C: unit cost of production
- P: sale price
- $D(\omega)$: random demand

Two-stage stochastic formulation of newsboy problem:

$$\min_{x, s(\omega) \geq 0} C \cdot x - \mathbb{E}_{\omega}[P \cdot s(\omega)]$$
$$\text{s.t. } s(\omega) \leq x$$
$$s(\omega) \leq D(\omega)$$

Extensions: salvage value, penalty for unserved demand

What is the trade-off of large/small value of x?

Example: Capacity Expansion Planning

$$\min_{x,y\geq 0} \sum_{i=1}^{n} (I_i \cdot x_i + \mathbb{E}_\omega[\sum_{j=1}^{m} C_i \cdot T_j \cdot y_{ij}(\omega)])$$

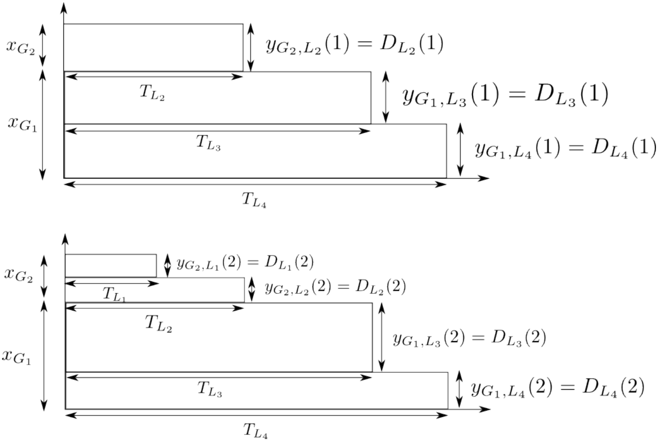
$$\text{s.t.} \sum_{i=1}^{n} y_{ij}(\omega) = D_j(\omega), j = 1, \ldots, m$$

$$\sum_{j=1}^{m} y_{ij}(\omega) \leq x_i, i = 1, \ldots n-1$$

- I_i, C_i: fixed/variable cost of technology i
- $D_j(\omega), T_j$: height/width of load block j
- $y_{ij}(\omega)$: capacity of i allocated to j
- x_i: capacity of i

Note: T_j independent of ω

Example: Capacity Expansion Planning - Graphical Illustration



- Note: T_j independent of ω
- How did we get rid of block 1 in first scenario ($\omega = 1$)?

Example: Hydro-Thermal Scheduling

Denote:

- q_t: hydro power
- p_t: thermal power
- C: marginal cost of thermal power plant
- D_t: demand
- E: storage limit in the dam
- x_t: content of dam at the *end* of a stage
- r_t: amount of rain during stage t

Hydro-thermal scheduling problem:

$$
\begin{aligned}
&\min C \cdot p_1 + \mathbb{E}_\omega[C \cdot p_2(\omega)] \\
&p_1 + q_1 \geq D_1 \\
&x_1 \leq x_0 + r_1 - q_1 \\
&x_1 \leq E \\
&p_2(\omega) + q_2(\omega) \geq D_2 \\
&q_2(\omega) \leq x_1 + r_2(\omega) \\
&p_1, q_1, x_1, p_2(\omega), q_2(\omega) \geq 0
\end{aligned}
$$

What is the trade-off?

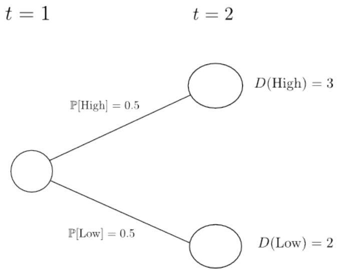
Scenario Tree

Scenario tree: graphical representation of uncertainty in stochastic programs

Characteristics of scenario tree

- Time *stages*
- Nodes/outcomes at each stage
- Transition probabilities from nodes/outcomes of stage t to nodes/outcomes of stage $t + 1$

Example: Newsboy - Scenario Tree



- $D(\omega) = \begin{cases} 2, & \omega = 1 \\ 3, & \omega = 2 \end{cases}$ with equal likelihood
- One node/outcome in stage 1, two nodes/outcomes in stage 2

Outline

Modeling Uncertainty in Multistage Stochastic Linear Programming (MSLP)

Recall multi-period decision making under uncertainty:

- S_t: state space at stage t
- A_t: action space at stage t
- The general framework of multi-period decision making under uncertainty defines $\Omega = (S_0, A_0) \times \ldots \times (S_{H-1}, A_{H-1}) \times S_H$

Multi-stage stochastic linear programming (MSLP):

- The above framework is unnecessarily general for MSLPs
- In MSLPs one defines a set S_t at each stage, and the sample space $\Omega = S_1 \times \ldots, S_H$ (note that in MSLP time counts from $t = 1$)
- We can then define the probability space $(\Omega, \mathcal{A}_H, \mathbb{P})$, where $\mathcal{A}_H = \mathcal{B}(\Omega_H)$, and $\mathbb{P}$ is a probability measure on $\mathcal{A}_H$
- $\mathbb{P}$ implies a probability measure for any $\mathcal{A}_t \subseteq \mathcal{B}(\Omega)$

Scenario Trees

In practice, nobody uses $(\Omega, \mathcal{A}_H, \mathbb{P})$ to describe uncertainty (too tedious, unnecessarily complex)

Scenario trees: compact, graphical representation of uncertainty in stochastic programming

- A tree (N, E) is an acyclic connected graph, where N denotes its nodes and E denotes its edges
- A **rooted tree** is a tree where there exists a unique node which is designated as the **root**
- A **scenario tree** is a rooted tree which consists of
  - *outcomes* (or *nodes*): history of random input $\xi_{[t]} = (\xi_1, \ldots, \xi_t)$ (each node characterized by a unique time index)
  - edges: transition probabilities from a history $\xi_{[t_1]}$ in stage t_1 to history $\xi_{[t_2]}$ in stage t_2, with $t_1 \leq t_2$

Scenario Tree Structure

- Root corresponds to $t = 1$
- **Ancestor** of a node $\xi_{[t]}$, $A(\xi_{[t]})$: *unique* adjacent node which is indexed by an earlier point in time

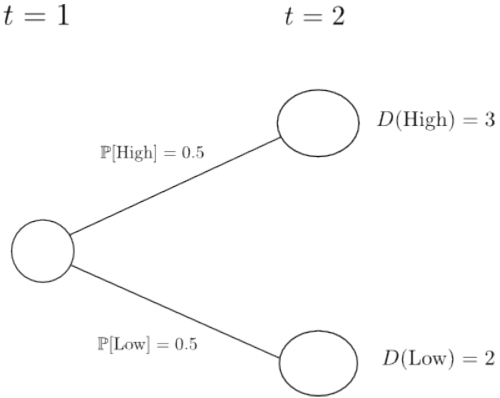
$$A(\xi_{[t]}) = \{\xi_{[t']} : (\xi_{[t']}, \xi_{[t]}) \in E\}$$

- **Children** or **descendants** of a node, $C(s_t)$: set of nodes that are adjacent to s_t and are indexed by a later moment in time

$$C(\xi_{[t]}) = \{\xi_{[t']} : (\xi_{[t]}, \xi_{[t']}) \in E\}$$

- Each node of stage t corresponds to a realization of a random vector $\xi_{[t]}$

Example: Newsboy Problem



Formal definition of probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$:

- Sample space: $\Omega = \{\text{High}, \text{Low}\}$
- Probability measure: $\mathbb{P}[\text{High}] = \mathbb{P}[\text{Low}] = 0.5$
- Random variable: $D(\text{Low}) = 3$, $D(\text{High}) = 2$

Lattice

A **lattice** is a scenario tree where every state $\xi_{[t-1]} \in \Xi_{[t-1]}$ has the same set of descendants in $\xi_{[t]} \in \Xi_t$

Lattices are more compact representations of uncertainty than scenario trees, typically employed in SDDP

Serial Independence

Define

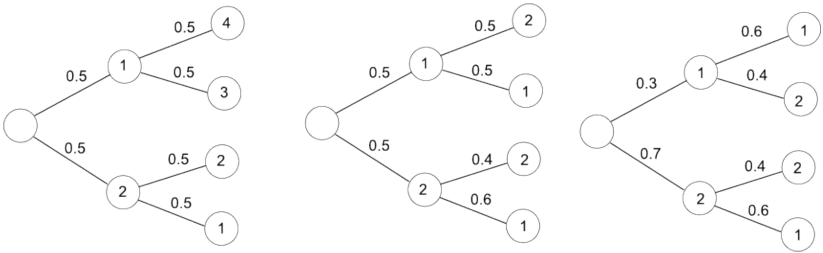
- $\xi_{[t-1]}(\omega) = (\xi_1(\omega), \ldots, \xi_{t-1}(\omega))$ as the history of realizations of the random input up to stage $t-1$
- $\Xi_{[t]} = \Xi_1 \times \ldots \times \Xi_t$ the set of possible paths from time period 1 to t

Serial independence: random disturbance $\xi_t(\omega)$ in each stage t has a probability distribution that does not depend on the history of the process, i.e. one can define a probability measure $p_t(i)$ at each stage t, such that

$$\mathbb{P}[\xi_t(\omega) = i | \xi_{[t-1]}(\omega)] = p_t(i), \forall \xi_{[t-1]} \in \Xi_{[t-1]}, i \in \Xi_{[t]}$$

Examples

Values on arcs indicate transition probabilities, values in nodes indicate realization of ξ_t



Which scenario tree(s) obey(s) serial independence

Outline

General Formulation

$$
\begin{aligned}
(MSLP): \quad & \min c_1^T x_1 + \mathbb{E}[c_2(\omega)^T x_2(\omega) + \cdots + c_H(\omega)^T x_H(\omega)] \\
& \text{s.t. } W_1 x_1 = h_1 \\
& T_1(\omega) x_1 + W_2(\omega) x_2(\omega) = h_2(\omega), \omega \in \Omega \\
& \vdots \\
& T_{t-1}(\omega) x_{t-1}(\omega) + W_t(\omega) x_t(\omega) = h_t(\omega), \omega \in \Omega \\
& \vdots \\
& T_{H-1}(\omega) x_{H-1}(\omega) + W_H(\omega) x_H(\omega) = h_H(\omega), \omega \in \Omega \\
& x_1 \geq 0, x_t(\omega) \geq 0, t = 2, \ldots, H
\end{aligned}
$$

Notation

- $c_t(\omega) \in \mathbb{R}^{n_t}$: cost coefficients
- $h_t(\omega) \in \mathbb{R}^{m_t}$: right-hand side parameters
- $W_t(\omega) \in \mathbb{R}^{m_t \times n_t}$: coefficients of $x_t(\omega)$
- $T_{t-1}(\omega) \in \mathbb{R}^{m_t \times n_{t-1}}$: coefficients of $x_{t-1}(\omega)$
- $x_t(\omega)$: set of state *and* action variables in period t
- Action space / feasible set of $x_t(\omega)$:

$$
\begin{aligned}
&T_{t-1}(\omega)x_{t-1}(\omega) + W_t(\omega)x_t(\omega) = h_t(\omega) \\
&x_t(\omega) \geq 0
\end{aligned}
$$

Formulation on a Scenario Tree

$$
\begin{aligned}
&\min c_1^T x_1 + \mathbb{E}[c_2(\omega_{[t]})^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_{[H]})^T x_H(\omega_{[H]})] \\
&\text{s.t. } W_1 x_1 = h_1 \\
&T_1(\omega_2)x_1 + W_2(\omega_{[2]})x_2(\omega_{[2]}) = h_2(\omega_{[2]}), \omega_{[2]} \in \Xi_{[2]} \\
&\vdots \\
&T_{t-1}(\omega_{[t]})x_{t-1}(\omega_{[t-1]}) + W_t(\omega_{[t]})x_t(\omega_{[t]}) = h_t(\omega_{[t]}), \omega_{[2]} \in \Xi_{[2]} \\
&\vdots \\
&T_{H-1}(\omega_{[H]})x_{H-1}(\omega_{[H]}) + W_H(\omega_{[H]})x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in \Xi_{[H]} \\
&x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \ldots, H
\end{aligned}
$$

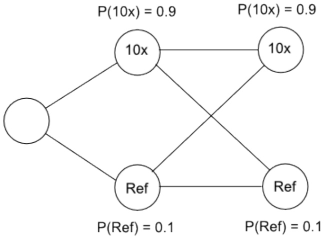
Note: random input ξ_t and decision vector x_t is indexed over history $\omega_{[t]} \in \Xi_{[t]}$

Formulation on Lattice

$$
\begin{aligned}
(MSLP): \quad & \min c_1^T x_1 + \mathbb{E}[c_2(\omega_2)^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_H)^T x_H(\omega_{[H]})] \\
& \text{s.t. } W_1 x_1 = h_1 \\
& T_1(\omega_2)x_1 + W_2(\omega_2)x_2(\omega_{[2]}) = h_2(\omega_2), \omega_{[2]} \in \Xi_{[2]} \\
& \vdots \\
& T_{t-1}(\omega_t)x_{t-1}(\omega_{[t-1]}) + W_t(\omega_t)x_t(\omega_{[t]}) = h_t(\omega_t), \omega_{[t]} \in \Xi_{[t]} \\
& \vdots \\
& T_{H-1}(\omega_H)x_{H-1}(\omega_{[H-1]}) + W_H(\omega_H)x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in \Xi_{[H]} \\
& x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \ldots, H
\end{aligned}
$$

Note: ξ_t is indexed over $\omega_t \in \Xi_t$, while x_t is indexed over $\omega_{[t]} \in \Xi_{[t]}$.

Example: Capacity Expansion - Scenario Tree



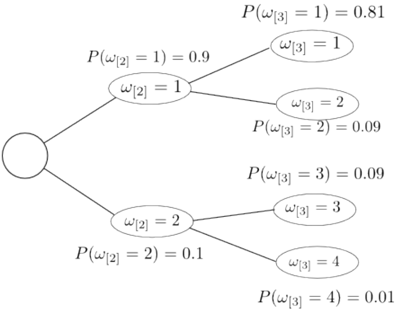


Table: Load duration curve for reference and 10x outcome

	Duration (hours)	Level (MW) Reference scenario	Level (MW) 10x wind scenario
Base load	8760	0-7086	0-3919
Medium load	7000	7086-9004	3919-7329
Peak load	1500	9004-11169	7329-10315

Example: Capacity Expansion - Technological Options

Technology	Fuel cost ($/MWh)	Inv cost ($/MWh)
Coal	25	16
Gas	80	5
Nuclear	6.5	32
Oil	160	2
DR	1000	0

Example: Capacity Expansion - Notation and Setup

Denote:

- $v_{it\omega_{[t]}}$: capacity of technology i constructed in period t
- $x_{it\omega_{[t]}}$: *total* amount of capacity of technology i available in period t
- $y_{ijt\omega_{[t]}}$: power allocation from technology i to load block j

Sequence of events:

1. Capacity $x_{i,t-1,\omega_{[t-1]}}$ available at the *end* of stage $t-1$ that can serve demand in t
2. Demand $D_{jt\omega_{[t]}}$ is observed
3. Construct new capacity $v_{it\omega_{[t]}}$

Example: Capacity Expansion - Model

Objective function:

$$\min_{x,v,y\geq 0} \sum_{i=1}^{n} I_i \cdot v_{i11}$$
$$+ \sum_{\omega_{[2]}=1}^{2} p_{\omega_{[2]}} (\sum_{i=1}^{n} I_i \cdot v_{i2\omega_{[2]}} + \sum_{i=1}^{n} \sum_{j=1}^{m} C_i \cdot T_j \cdot y_{ij2\omega_{[2]}})$$
$$+ \sum_{\omega_{[3]}=1}^{4} p_{\omega_{[3]}} (\sum_{i=1}^{n} I_i \cdot v_{i3\omega_{[3]}} + \sum_{i=1}^{n} \sum_{j=1}^{m} C_i \cdot T_j \cdot y_{ij3\omega_{[3]}})$$

Note: first stage involves *only* investment decision

Supply equals demand (enforced only for $t > 1$):

$$\sum_{i=1}^{n} y_{ijt\omega_{[t]}} = D_{jt\omega_{[t]}}, j \in \{1, \ldots, m\}, t \in \{2, \ldots, 3\}$$
$$\omega_{[2]} \in \{1, 2\}, \omega_{[3]} \in \{1, \ldots, 4\}$$

Example: Capacity Expansion - Model

Investment dynamics:

$$x_{i2\omega_{[2]}} = x_{i11} + v_{i2\omega_{[2]}}, i \in \{1, \ldots, n-1\}, \omega_{[2]} \in \{1, 2\}$$
$$x_{i3\omega_{[3]}} = x_{i21} + v_{i3\omega_{[3]}}, i \in \{1, \ldots, n-1\}, \omega_{[3]} \in \{1, 2\}$$
$$x_{i3\omega_{[3]}} = x_{i22} + v_{i3\omega_{[3]}}, i \in \{1, \ldots, n-1\}, \omega_{[3]} \in \{3, 4\}$$

Technology capacity constraints:

$$\sum_{j=1}^{m} y_{ij2\omega_{[2]}} \leq x_{i11}, i \in \{1, \ldots n-1\}, \omega_{[2]} \in \{1, 2\}$$
$$\sum_{j=1}^{m} y_{ij3\omega_{[3]}} \leq x_{i21}, i \in \{1, \ldots n-1\}, \omega_{[3]} \in \{1, 2\}$$
$$\sum_{j=1}^{m} y_{ij3\omega_{[3]}} \leq x_{i22}, i \in \{1, \ldots n-1\}, \omega_{[3]} \in \{3, 4\}$$

Does this model obey block separability?

Example: Capacity Expansion - Optimal Solution

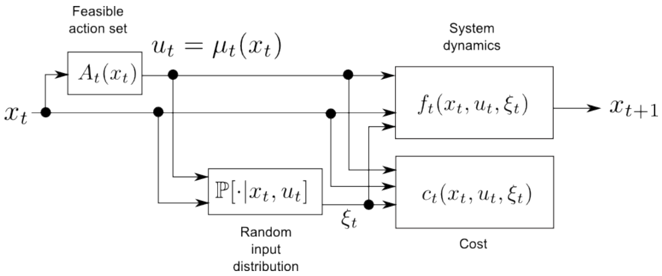
Optimal expansion plan:

- Coal, period 1: 2986 MW
- Nuclear, period 1: 7329 MW
- Oil, period 1: 854 MW
- Period 2: nothing (!)

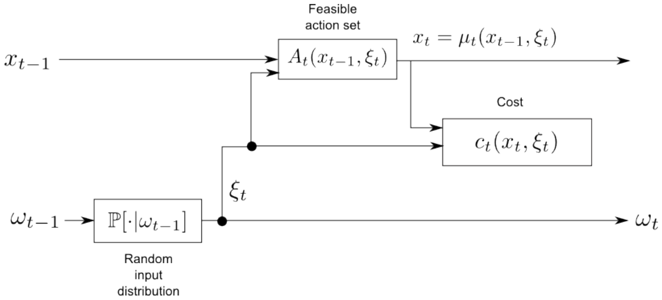
Why is it optimal to invest *only* in period 1?

Outline

Multistage Decision Making Under Uncertainty



Multistage Stochastic Linear Programming



Differences between the two block diagrams:

- In MSLP, decision maker *first* observes the realization of uncertainty, ξ_t, *then* decides x_t
- System state which is propagated from one period to the next: vector x_t *and* node in the scenario tree ω_t
- Feasible action set A_t depends on x_{t-1} *and* ξ_t

Block Separability

Feasible action set in stage t:

$$T_{t-1}(\omega_t)x_{t-1}(\omega_{[t]}) + W_t(\omega_t)x_t(\omega_{[t]}) = h_t(\omega_t), \omega_t \in \Xi_{[t]}$$

Block separability occurs when these constraints can be written in the following form:

$$T^{xx}_{t-1}(\omega_t)x_{t-1}(\omega_{[t-1]}) + W^{xx}_t(\omega_t)x_t(\omega_{[t]}) = h^{xx}_t(\omega_t), \omega_t \in \Xi_{[t]}$$
$$T^{xu}_{t-1}(\omega_t)x_{t-1}(\omega_{[t-1]}) + W^{xu}_t(\omega_t)u_t(\omega_t) = h^{xu}_t(\omega_t), \omega_t \in \Xi_{[t]}$$

So what?

- Decision variables u_t do not need to be propagated forward
- Computational benefits: SDDP does not scale well for large dimension of x_t vector, can handle high-dimensional action vectors efficiently

Application of Dynamic Programming in MSLP

Q-**function** in final period:

$$\begin{aligned} Q_H(x_{H-1}, \xi_H) = \quad & \min_{x_H} c_H(\omega_H)^T x_H \\ & \text{s.t. } T_{H-1}(\omega_H) x_{H-1} + W_H(\omega_H) x_H = h_H(\omega_H) \\ & x_H \geq 0 \end{aligned}$$

Value function in final period:

$$V_H(x_{H-1}, \omega_{H-1}) = \mathbb{E}_{\xi_H}[Q_H(x_{H-1}, \xi_H) | \omega_{H-1}]$$

Proceeding recursively, *Q*-function in stage t:

$$\begin{aligned} Q_t(x_{t-1}, \xi_t) = \quad & \min_{x_t} c_t(\omega_t)^T x_t + V_{t+1}(x_t, \omega_t) \\ & \text{s.t. } T_{t-1}(\omega_t) x_{t-1} + W_t(\omega_t) x_t = h_t(\omega_t) \\ & x_t \geq 0 \end{aligned}$$

Value function in stage t:

$$V_t(x_{t-1}, \omega_{t-1}) = \mathbb{E}_{\xi_t}[Q_t(x_{t-1}, \xi_t) | \omega_{t-1}]$$

Notational Convention

- Note the different notation for node of the lattice (ω_t) and realization of uncertainty (ξ_t)
- It is *only* necessary to propagate (x_t, ω_t) forward, *not* (x_t, ξ_t)
- The notation $V_t(x_{t-1}, \omega_{t-1})$ emphasizes how value functions are stored by SDDP
- The notation $Q_t(x_{t-1}, \xi_t)$ is the conventional notation used in stochastic programming, but Q-functions are not explicitly stored in SDDP

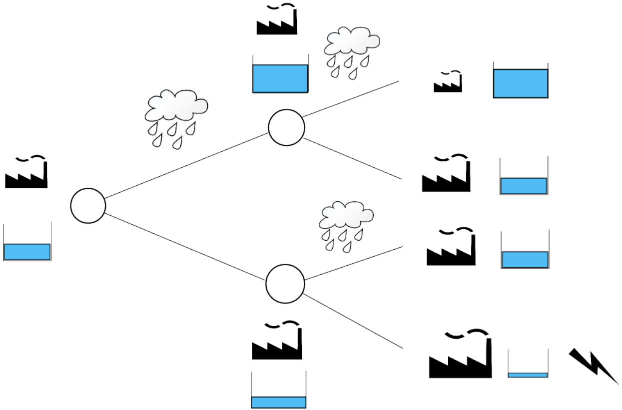
Application of Dynamic Programming to 2-Stage Stochastic Linear Programming

$$
\begin{aligned}
Q_t(x_{t-1}, \omega_t) \quad = \quad & \min c_t(\omega_t)^T x_t + V_{t+1,\omega_t}(x_t) \\
& \text{s.t. } W_t(\omega_t) x_t = h_t(\omega_t) - T_{t-1}(\omega_t) x_{t-1} \\
& x_t \geq 0
\end{aligned}
$$

Proceed backwards until:

$$
\begin{aligned}
& \min c_1^T x_1 + V_2(x_1) \\
& \text{s.t. } W_1 x_1 = h_1 \\
& x_1 \geq 0
\end{aligned}
$$

Example: Hydrothermal Scheduling



Example: Hydrothermal Scheduling

Consider the following hydro-thermal system:

- 3 periods
- Demand in each period: 1000 MW
- Marginal cost of thermal generators: 25 $/MWh
- Max production of thermal generators: 500 MW
- Marginal cost of lost load: 1000 $/MWh
- Rainfall: *independent* identically distributed, uniformly on $[0, 1000]$ MW, denote density function as $f : \mathbb{R} \to \mathbb{R}$

Example: Hydrothermal Scheduling

Denote

- p: thermal production
- q: hydro production
- l: unserved demand
- x_2: stored hydro energy at *beginning* of period 2

$$
\begin{aligned}
Q_3(x_2, R_3) = \quad & \min 1000 \cdot l + 25 \cdot p \\
& \text{s.t. } l + p + q \geq 1000 \\
& p \leq 500 \\
& q \leq x_2 + R_3 \\
& l, p, q \geq 0
\end{aligned}
$$

Q function of period 3:

$$Q_3(x_2, R_3) = \begin{cases} 0, x_2 + R_3(\omega) \geq 1000 \\ 25 \cdot (1000 - (x_2 + R_3(\omega))), 500 \leq x_2 + R_3(\omega) < 1000 \\ 500 \cdot 25 + 1000 \cdot (500 - (x_2 + R_3(\omega))), 0 \leq x_2 + R_3(\omega) < 500 \end{cases}$$

Value function of period 3:

$$\begin{aligned} & V_3(x_2) = \mathbb{E}_{R_3}[Q_3(x_2, R_3)] \\ & = \mathbb{P}[R_3(\omega) \geq 1000 - x_2] \cdot 0 \\ & + \int_{r=500-x_2}^{1000-x_2} (25 \cdot (1000 - r - x_2)) f(r) dr \\ & + \int_{r=0}^{500-x_2} (500 \cdot 25 + 1000 \cdot (500 - r - x_2)) f(r) dr \\ & = \begin{cases} 0, & x_2 \geq 1000 \\ 12500 - 25 \cdot x_2 + 0.0125 \cdot x_2^2, & 500 \leq x_2 < 1000 \\ 134375 - 512.5 \cdot x_2 + 0.5 \cdot x_2^2, & 0 \leq x_2 < 500 \end{cases} \end{aligned}$$

Note:

- V_3 is convex
- V_3 is *not* a piecewise linear function of x_2

Q_2 can be computed as:

$$
\begin{aligned}
& Q_2(x_1, R_2) = \min 1000 \cdot l + p + V_3(x_2) \\
& \text{s.t. } l + p + q \geq 1000, p \leq 500 \\
& x_2 = x_1 - q + R_2(\omega) \\
& l, p, q, x_2 \geq 0
\end{aligned}
$$

Q_2 yields V_2, V_2 yields Q_1, ...