Performance of Stochastic Programming Solutions

Operations Research

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Performance of Stochastic Programming Solutions

Two-Stage Stochastic Linear Programs

$$\min z = c^T x + \mathbb{E}_\omega[\min q(\omega)^T y(\omega)]$$
$$\text{s.t. } Ax = b$$
$$T(\omega)x + W(\omega)y(\omega) = h(\omega)$$
$$x \geq 0, y(\omega) \geq 0$$

- First stage decisions $x \in \mathbb{R}^{n_1}$, $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{m_1 \times n_1}$
- For a given realization ω, second-stage data are $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$, $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$, $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$
- All random variables of the problem are assembled in a single random vector $\xi^T(\omega) = (q(\omega)^T, h(\omega)^T, T_{1\cdot}(\omega), \ldots, T_{m_2\cdot}(\omega), W_{1\cdot}(\omega), \ldots, W_{m_2\cdot}(\omega))$

Motivation

Is it worth solving a stochastic program?

- How well could we do if we knew the future?
- How well could we do with a simpler model (e.g. expected value problem)?

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Notation

$$z(x, \xi) = c^T x + Q(x, \omega) + \delta(x|K_1)$$
$$Q(x, \xi) = \min_y \{q(\omega)^T y | W(\omega)y = h(\omega) - T(\omega)x\}$$

- What is the interpretation of $z(x, \xi)$?
- Define $K_1 = \{x | Ax = b, x \geq 0\}$ as the set of feasible first-stage decisions
- Define $K_2(\omega) = \{x | \exists y : W(\omega)y = h(\omega) - T(\omega)x\}$ as the set of first-stage decisions that have a feasible reaction in the second stage for $\omega \in \Omega$
- It can be that $z(x, \xi) = +\infty$ (if $x \notin K_1 \cap K_2(\omega)$)
- It can be that $z(x, \xi) = -\infty$ (unbounded below)

Wait-and-See, Here-and-Now

- The **wait-and-see** value is the expected value of reacting with perfect foresight $x^{\star}(\xi)$ to ξ:

$$WS = \mathbb{E}[\min_{x} z(x, \xi)]$$

$$\mathbb{E}[z(x^{\star}(\xi), \xi)]$$

- The **here-and-now** value is the expected value of the recourse problem:

$$SP = \min_{x} \mathbb{E}[z(x, \xi)]$$

- We have swapped min and $\mathbb{E}$. What's the difference?
- Which one is more difficult to compute?

Expected Value of Perfect Information (EVPI)

The **expected value of perfect information** is the difference between the two solutions:

$$EVPI = SP - WS$$

Interpretation: value of a perfect forecast for the future

Example: Capacity Expansion Planning

Technology	Fuel cost ($/MWh)	Inv cost ($/MWh)
Coal	25	16
Gas	80	5
Nuclear	6.5	32
Oil	160	2
DR	1000	0

Table: Probability of (i) reference load duration curve: 10%, (ii) 10x wind scenario: 90%.

	Duration (hours)	Level (MW) Reference scenario	Level (MW) 10x wind scenario
Base load	8760	0-7086	0-3919
Medium load	7000	7086-9004	3919-7329
Peak load	1500	9004-11169	7329-10315

Technology	SP solution	Reference	10x wind	EV solution
Coal	5085	1918	3410	4235
Gas	1311	2165	2986	3261
Nuclear	3919	7086	3919	2905
Oil	854	0	0	0

$$SP = 340316 \text{ \$/h}$$

$$z(x^\star(''Ref''), ''Ref'') = 382288 \text{ \$/h}$$

$$z(x^\star(''10x''), ''10x'') = 329383 \text{ \$/h}$$

$$WS = 334673 \text{ \$/h}$$

$$EVPI = 5643 \text{ \$/h} = 1.7\% \cdot SP$$

Note: wait-and-see model never chooses oil

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Expected Value Problem

Expected (or mean) value problem:

$$EV = \min_x z(x, \bar{\xi}), \bar{\xi} = \mathbb{E}[\xi]$$

Expected value solution $x^\star(\bar{\xi})$: optimal solution of expected value problem

Value of the Stochastic Solution

The **expected value of using the EV solution** measures the performance of $x^\star(\bar{\xi})$ (optimal second-stage reactions given $x^\star(\bar{\xi})$):

$$EEV = \mathbb{E}[z(x^\star(\bar{\xi}), \xi)]$$

The **value of the stochastic solution** is

$$VSS = EEV - SP$$

- Which one is easier to compute: WS, SP, or EEV? Which one is harder?
- What can we say about VSS if $x^\star(\xi)$ is independent of ξ?

Example: Capacity Expansion Planning

Table: Optimal investment and fixed cost for the stochastic program and the expected value problem.

	SP investment (MW)	EV investment (MW)	SP fixed cost ($/h)	EV fixed cost ($/h)
Coal	5085	3261	81360	52176
Gas	1311	2905	6,555	14525
Nuclear	3919	4235	125408	135520
Oil	854	0	1708	0
Total	11169	10401	215031	202221

Example: Capacity Expansion Planning

Table: Variable cost for the SP and EV models.

	SP var cost ($/h)	EV var cost ($/h)
Block 1	25473	25473
Block 2	64858	60070
Block 3	4854	4854
Block 4	9799	29209
Block 5	17960	17959
Block 6	2340	13268
Total	125285	150834

- $EEV = 12739$ $/h
- Investment cost of *EV* solution is lower than *SP* solution
- *EV* investment cannot serve peak demand in "Ref" scenario

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Crystal Ball

- For every ξ, we have $z(x^\star(\xi), \xi) \leq z(x^\star, \xi)$ where $x^\star$ is the optimal solution to the stochastic program
- Taking expectations on both sides, $WS \leq SP$

Interpretation: we can do better if we have a crystal ball (i.e. we know the future in advance)

Lazy Solution

- $x^\star$ is the optimal solution of

$$\min_x \mathbb{E}[z(x,\xi)]$$

- $x^\star(\bar{\xi})$ is a solution (not necessarily optimal), therefore

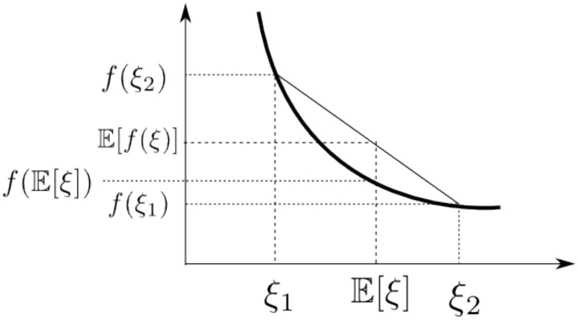
$$\min_x \mathbb{E}[z(x,\xi)] = SP \leq EEV = \mathbb{E}[z(x^\star(\bar{\xi}),\xi)]$$

Interpretation: we do worse when we are lazy (i.e. when we do not account for uncertainty explicitly)

Would anything change if some of the x, y were integer?

Jensen's Inequality

Suppose f is convex and ξ is a random variable, then $f(\mathbb{E}[\xi]) \leq \mathbb{E}[f(\xi)]$



Lazy and a Liar!

Suppose c, W, T are independent of ω (i.e. , $\xi = h$): then $EV \leq WS$

- We will show that $z(x, h)$ is jointly convex in (x, h)
- We know that $f(\xi) = \min_x z(x, \xi)$ is convex in ξ
- From Jensen's inequality, we have $\mathbb{E}[f(\xi)] \geq f(\mathbb{E}[\xi])$

Interpretation: EV (the lazy solution) is a biased estimate of expected cost. Is it optimistic, or pessimistic?

Proof that $z(x, h)$ is convex in (x, h)

- Consider x_1, x_2 and $\lambda \in (0, 1)$. Without loss of generality, assume $Ax_1 = b$, $Ax_2 = b$, $x_1, x_2 \geq 0$.
- $z(x_i, h_i) = c^T x_i + q^T y_i$, where
  $y_i = \min\{q^T y | Wy = h_i - Tx_i, y \geq 0\}, i = \{1, 2\}$
- $z(\lambda x_1 + (1-\lambda)x_2, \lambda h_1 + (1-\lambda)h_2) = c^T(\lambda x_1 + (1-\lambda)x_2) + q^T y_\lambda$, where
  $y_\lambda = \min\{q^T y | Wy = \lambda h_1 + (1-\lambda)h_2 - T(\lambda x_1 + (1-\lambda)x_2), y \geq 0\}$
- $\lambda y_1 + (1-\lambda)y_2$ is a feasible solution for
  $\min\{q^T y | Wy = \lambda h_1 + (1-\lambda)h_2 - T(\lambda x_1 + (1-\lambda)x_2), y \geq 0\}$.
  Therefore, we have $q^T y_\lambda \leq \lambda q^T y_1 + (1-\lambda)q^T y_2$.
- It follows that
  $z(\lambda x_1 + (1-\lambda)x_2, \lambda h_1 + (1-\lambda)h_2) \leq \lambda z(x_1, h_1) + (1-\lambda)z(x_2, h_2)$

Example: Capacity Expansion Planning

Does the cap ex problem satisfy the assumptions of slide 20?

For the capacity expansion problem:

$$WS = EV = 334674 \text{ \$/h}$$

Exercise: show that $EV = WS$ holds in general for the two-stage stochastic capacity expansion problem with demand uncertainty

Counter-Example: $EV > WS$

Consider the following problem:

$$\begin{aligned} &\min_{x \geq 0} 2x + \mathbb{E}_\xi[\xi \cdot y] \\ &\text{s.t. } y \geq 1 - x \\ &y \geq 0 \end{aligned}$$

where $\mathbb{P}[\xi = 1] = 3/4$, $\mathbb{P}[\xi = 3] = 1/4$

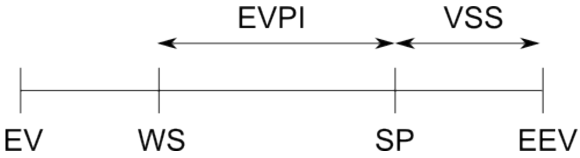
Does this problem satisfy the assumptions of slide 20?

- Optimal second-stage decision: $y = 1 - x$ if $1 - x \geq 0$, $y = 0$ otherwise
- Trade-off: by increasing x we can push y to lower values, but incur certain cost $2x$
- For $\bar{\xi} = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$ we have $\{\min 2x + \frac{3}{2}y | y \geq 1 - x, x \geq 0, y \geq 0\}$
- Optimal solution: $x^\star(\bar{\xi}) = 0$, $y = 1$ with $EV = \frac{3}{2}$
- To compute WS, note that for $\xi = 1$ the optimal first-stage decision is $x = 0$, with cost of 1, while for $\xi = 3$ the optimal first-stage decision is $x = 1$, with cost of 2: $WS = \frac{3}{4} + \frac{1}{4} \cdot 2 = \frac{5}{4} < EV$

Summary

We have established that

- $VSS \geq 0$, $EVPI \geq 0$
- $VSS \leq EEV - EV$, $EVPI \leq EEV - EV$
- If $EEV - EV = 0$ then $VSS = 0$, $EVPI = 0$ (for example, if $x^{\star}(\xi)$ independent of ξ - this is rare)



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Computing *EV*, *SP*, *WS*, *EEV*

- Computing *EV*: single linear program
- Computing two-stage *SP*: (multi-cut) L-shaped method
- Computing multi-stage *SP*: nested decomposition, SDDP
- *EEV* and *WS*: simulation

Notes:

- Generalization of *WS* to multiple stages is fairly obvious
- Generalization of *EEV* to multiple stages is not obvious
- Consider discretization of n random variables at d values each, exact computation of *EEV* and *WS* requires solving d^n linear programs

Estimating *WS* and *EEV*

Estimation of *WS* and *EEV* through sample mean approximation:

- For $i = 1, \ldots, K$
  - Sample $\xi_i = \xi(\omega_i)$
  - Compute $x^\star(\bar{\xi})$
  - Compute $WS_i = z(x^\star(\xi_i), \xi_i)$ and $EEV_i = c^T x^\star(\bar{\xi}) + Q(x^\star(\bar{\xi}), \xi_i)$
- Estimate $\bar{WS} = \frac{1}{K} \sum_{i=1}^{K} WS_i$ and $E\bar{E}V = \frac{1}{K} \sum_{i=1}^{K} EEV_i$

Central Limit Theorem

Suppose $\xi(\omega)$ is continuous, does this complicate the computation of EV, SP, EEV and WS?

Central limit theorem: Suppose $\{X_1, X_2, ...\}$ is a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $Var[X_i] = \sigma^2 < \infty$. Then as n approaches infinity, $\sqrt{n}(S_n - \mu)$ converge in distribution to a normal $N(0, \sigma^2)$:

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \xrightarrow{d} N(0,\ \sigma^2).$$

Can we use the CLT? What would the X_i be in our case?

Example: Slow Convergence of Sample Average Approximation

The cost C of operating a facility is

- $C(N) = 1$ under normal operations, $f(N) = 0.9$
- $C(E) = 100$ under emergency operations, $f(E) = 0.1$

$$\mu = 0.1 \cdot 100 + 0.9 \cdot 1 = 10.9$$

$$\sigma = \sqrt{0.9 \cdot (1 - 10.9)^2 + 0.1 \cdot (100 - 10.9)^2} = 29.7$$

- Rare outcome (1 out of 10 samples) influences expected value calculation since it contributes by $\frac{0.1 \cdot 100}{10.9} = 91.7\%$ to expected value
- From central limit theorem, in order to get estimate of $\mathbb{E}[C]$ to be within 5% with 95.4% confidence, we need $2\frac{\sigma}{\sqrt{n}} = 0.05\mu$, from which $n = 11879$!

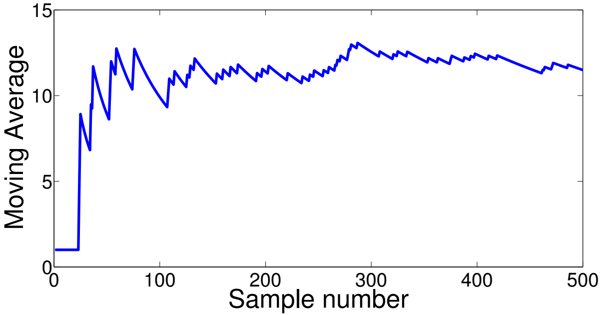

Figure: A sample of the evolution of the moving average $\frac{1}{n}\sum_{i=1}^{n} C(\omega_i)$ where ω_i denotes the outcome of sample i.

Note sensitivity of sample average to emergency outcome

Importance Sampling

Suppose we wish to estimate $\mathbb{E}[C(\omega)]$, where ω is distributed according to $f(\omega)$

- Sample average pulls samples ω_i according to distribution $f(\omega)$ and estimates $\mathbb{E}[C(\omega)]$ with $\sum_{i=1}^{N} \frac{1}{N} C(\omega_i)$
- **Importance sampling** pulls samples ω_i according to distribution $g(\omega) = \frac{f(\omega) \cdot C(\omega)}{\mathbb{E}[C]}$ and estimates $\mathbb{E}[C(\omega)]$ with $\sum_{i=1}^{N} \frac{1}{N} \frac{f(\omega_i) \cdot C(\omega_i)}{g(\omega_i)}$

Motivation of Importance Sampling

Note that $\mathbb{E}[C(\omega)] = \int_\Omega C(\omega) \cdot f(\omega) d\omega = \int_\Omega \frac{C(\omega) \cdot f(\omega)}{g(\omega)} g(\omega) d\omega$

- The random variable $\frac{C(\omega) \cdot f(\omega)}{g(\omega)}$, which is distributed according to $g(\omega)$, also has expectation $\mathbb{E}[C]$
- Which $g(\omega)$ *minimizes* the variance of this new random variable?

$$g(\omega) = \frac{C(\omega) \cdot f(\omega)}{\mathbb{E}[C]}$$

  Any sample evaluates to $\mathbb{E}[C]$!
- We cheated: $g(\omega)$ requires knowledge of $\mathbb{E}[C]$, which is what we are estimating
- But we learned something: pull samples according to contribution to expected value, $\frac{C(\omega) \cdot f(\omega)}{\mathbb{E}[C]}$. Even if we do not know $\mathbb{E}[C]$, we can *approximate* it.

Back to the Example

- Problem: rare 'bad' outcome had the greatest influence on expected value
- Remedy: redefine distribution so that we observe 'bad' outcome earlier, then adjust our expected value calculations in order to unbias result

$$g(\omega_1) = \frac{f(\omega_1) \cdot C(\omega_1)}{\mathbb{E}[C]} = \frac{0.9 \cdot 1}{10.9} = \frac{0.9}{10.9}$$
$$g(\omega_2) = \frac{f(\omega_2) \cdot C(\omega_2)}{\mathbb{E}[C]} = \frac{0.1 \cdot 100}{10.9} = \frac{10}{10.9}$$

Estimates from sampling ω_1, ω_2 are constant and equal to $\mathbb{E}[C]$:

$$C(\omega_1) \cdot \frac{f(\omega_1)}{g(\omega_1)} = 1 \cdot \frac{0.9}{\frac{0.9}{10.9}} = 10.9$$
$$C(\omega_2) \cdot \frac{f(\omega_2)}{g(\omega_2)} = 100 \cdot \frac{0.1}{\frac{10}{10.9}} = 10.9$$