The DC Optimal Power Flow

Quantitative Energy Economics

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Transmission Constraints

Lines can carry a limited amount of power

- Thermal limits
- Stability limits
- Voltage drop limits

Kirchhoff voltage and current laws

- Non-linear mapping: power injection in buses $\rightarrow$ power flow in lines
- We will linearize these

Optimal power flow problem (OPF): Maximize welfare (minimize cost) subject to Kirchhoff laws + transmission limits

Network Representation

Transmission system is represented as a directed graph

- N: set of nodes
- K: set of lines (denoted by $k = (m, n)$)
- G_n: set of generators located in node n, $G = \cup_{n \in N} G_n$
- L_n: set of loads located in node n, $L = \cup_{n \in N} L_n$

Two Equivalent Models

Decisions:

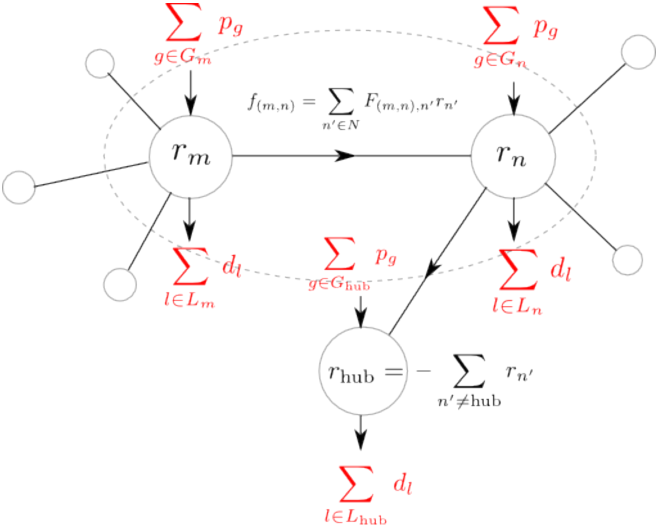
- p_g: amount of power produced by generator g
- d_l: amount of power consumed by load l

Two *equivalent* models, depending on system state and input data:

- Model 1
  - System state: nodal injections
  - Input data: power transfer distribution factors (depend on physical characteristics of lines)
- Model 2
  - System state: nodal phase angles
  - Input data: reactance (depend on physical characteristics of lines)

Table of Contents

Model 1: Power Transfer Distribution Factors



Net Injection

Hub node: reference node that "absorbs" all injections

Injection r_n: amount of power shipped from node n to the hub

$$r_n = \sum_{g \in G_n} p_g - \sum_{l \in L_n} d_l$$

Not amount of power flowing over line connecting n and hub

Conservation of energy:

$$\sum_{n \in N} r_n = 0$$

Power Flows

Power transfer distribution factor (PTDF) (F_{kn}): amount of power flowing on line k as a result of shipping 1 MW from n to hub

- $F_{\text{hub},n} = 0$
- PTDF: input data, depend on physical characteristics of lines
- PTDF depend on choice of hub
- Flow f_k is

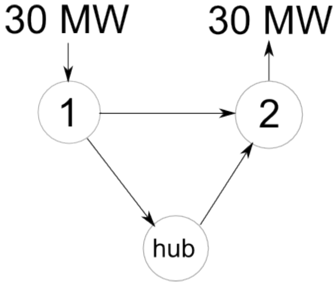
$$f_k = \sum_{n \in N} F_{kn} r_n$$

- Flow can be positive or negative (interpretation?)
- T_k: limit on power that each line can carry

$$-T_k \leq f_k \leq T_k$$

Example

All lines have identical electrical characteristics



1. $F_{1-2,1} = ?$, $F_{1-2,2} = ?$
2. Express shipment of 30 MW from 1 to 2 as transaction through hub
3. Compute flow f_{1-2} from steps 1, 2
4. Note: r_1 and $f_{1-\text{hub}}$ are *different*

The OPF Using PTDFs

$$\max \sum_{l \in L} \int_0^{d_l} MB_l(x)dx - \sum_{g \in G} \int_0^{p_g} MC_g(x)dx$$

$$(\lambda_k^+) : \quad f_k \leq T_k$$

$$(\lambda_k^-) : \quad -f_k \leq T_k$$

$$(\psi_k) : \quad f_k - \sum_{n \in N} F_{kn} r_n = 0$$

$$(\rho_n) : \quad r_n - \sum_{g \in G_n} p_g + \sum_{l \in L_n} d_l = 0$$

$$(\phi) : \quad \sum_{n \in N} r_n = 0$$

$$p_g, d_l \geq 0$$

Optimal Solution

Denote P_g, D_l as maximum production/consumption of generators/loads (imposed through domain of objective function)

There exists a threshold ρ_n for all n such that:

- If $0 < p_g < P_g$, then $\rho_n = MC_g(p_g)$. If $0 < d_l < D_l$, then $\rho_n = MB_l(d_l)$.
- If $p_g = P_g$, then $\rho_n \geq MC_g(P_g)$. If $d_l = D_l$, then $\rho_n \leq MB_l(D_l)$.
- If $p_g = 0$, then $\rho_n \leq MC_g(0)$. If $d_l = 0$, then $\rho_n \geq MB_l(0)$.

Proof: KKT conditions

$$0 \leq p_g \perp MC_g(p_g) - \rho_{n(g)} \geq 0$$

$$0 \leq d_l \perp -MB_l(d_l) + \rho_{n(l)} \geq 0$$

- $n(g)$: node where generator g is located
- $n(l)$: node where load l is located

Sensitivity

Helpful in understanding transmission pricing

- ϕ: marginal change in welfare from marginal increase in production/marginal decrease in consumption
- λ_k^+ and λ_k^-: marginal impact of increasing line capacity
- ρ_n: marginal impact of marginal increase of consumption/decrease of generation in node n (what if demand is inelastic?)

What sign do we expect for these dual variables?

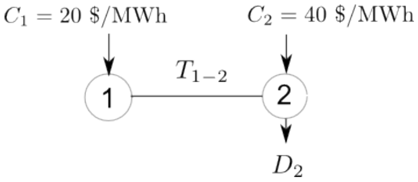
Components of ρ_n

Useful identity for computing prices:

$$\rho_n = -\phi + \sum_{k \in K} F_{kn} \lambda_k^- - \sum_{k \in K} F_{kn} \lambda_k^+$$

Proof: KKT conditions

Example

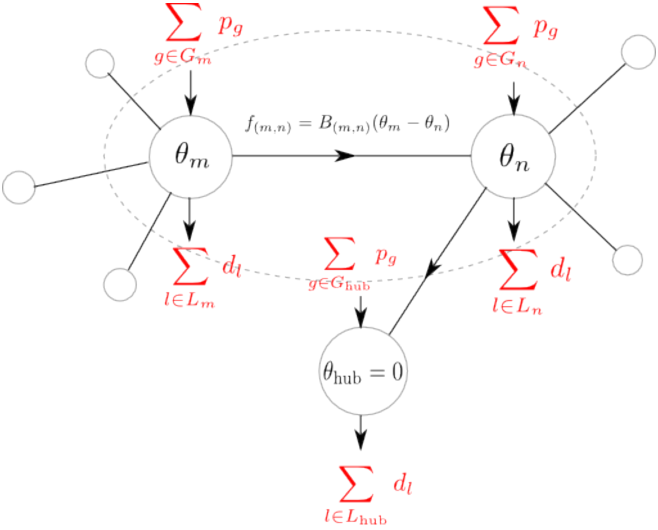


- Case 1
  - D_2 = 50 MW, T_{1-2} unlimited
  - $\rho_1 = \rho_2 = 20$ \$/MWh
- Case 2
  - D_2 = 50 MW, $T_{1-2} = 50$ MW
  - $\rho_1 = 20$ \$/MWh, 20 \$/MWh $\leq \rho_2 \leq$ 40 \$/MWh
- Case 3
  - D_2 = 60 MW, $T_{1-2} = 50$ MW
  - $\rho_1 = 20$ \$/MWh, $\rho_2 = 40$ \$/MWh

Can you explain multiplier values?

Table of Contents

Model 2: Reactance



Power Flows

- Reactance: input data, depends on physical characteristics of lines
- Independent on choice of hub
- Flow f_k is

$$f_{(m,n)} = B_{(m,n)}(\theta_m - \theta_n)$$

- Translation of θ results in identical flows, fix $\theta_{\text{hub}} = 0$
- Conservation of energy:

$$\sum_{g \in G_n} p_g + \sum_{k=(\cdot,n)} f_k = \sum_{k=(n,\cdot)} f_k + \sum_{l \in L_n} d_l.$$

- Input data is independent of network topology: transmission line investment, transmission line outages

The OPF Using Reactance

$$\max \sum_{l \in L} \int_0^{d_l} MB_l(x)dx - \sum_{g \in G} \int_0^{p_g} MC_g(x)dx$$

$$(\rho_n): \quad -\sum_{g \in G_n} p_g - \sum_{k=(\cdot,n)} f_k + \sum_{l \in L} d_l + \sum_{k=(n,\cdot)} f_k = 0$$

$$(\gamma_k): \quad f_k - B_k(\theta_m - \theta_n) = 0, k = (m, n)$$

$$(\lambda_k^+): \quad f_k \le T_k$$

$$(\lambda_k^-): \quad -f_k \le T_k$$

$$p_g, d_l \ge 0$$