Nested Decomposition

Operations Research

Anthony Papavasiliou

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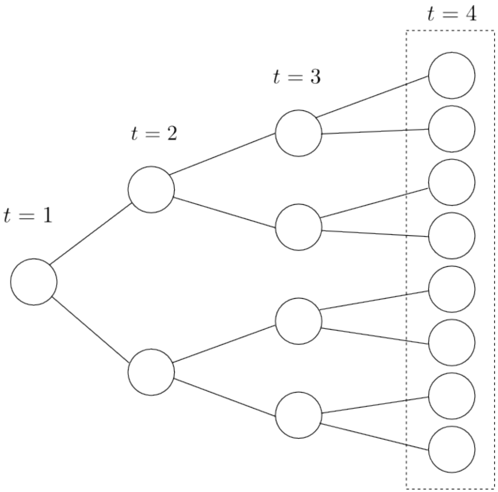
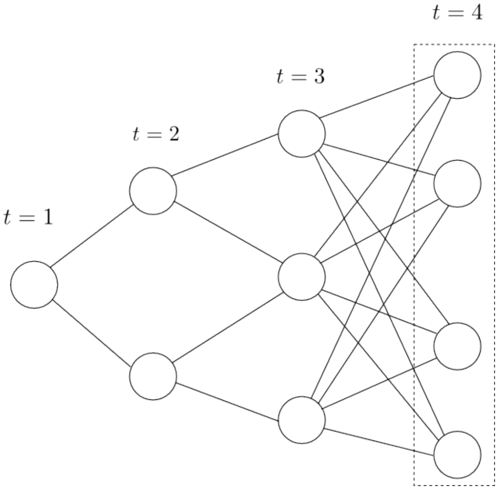
Scenario Tree



Lattices



Scenario Tree and Lattice Notation

- Dashed line marks the set Ξ_t
- $\Xi_{[t]} = \Xi_1 \times \ldots \times \Xi_t$
- Each node $\xi_{[t]} \in N$ is associated with a history of realizations of the stochastic input, $\xi_{[t]}$, and a probability of realization
- Each edge $(\xi_{[t_1]}, \xi_{[t_2]}) \in E$ is associated with a non-zero transition probability $\mathbb{P}[\xi_{[t_2]} | \xi_{[t_1]}]$, $t_2 > t_1$
- In the following, $c_{t,\omega}$ is used interchangeably for random variables, random vectors, and random matrices

Multi-Stage Stochastic Linear Programming on a Lattice

$$\min c_1^T x_1 + \mathbb{E}[c_2(\omega_2)^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_H)^T x_H(\omega_{[H]})]$$

$$\text{s.t. } W_1 x_1 = h_1$$

$$T_1(\omega_2)x_1 + W_2(\omega_2)x_2(\omega_{[2]}) = h_2(\omega_2), \omega_{[2]} \in \Xi_{[2]}$$

$$\vdots$$

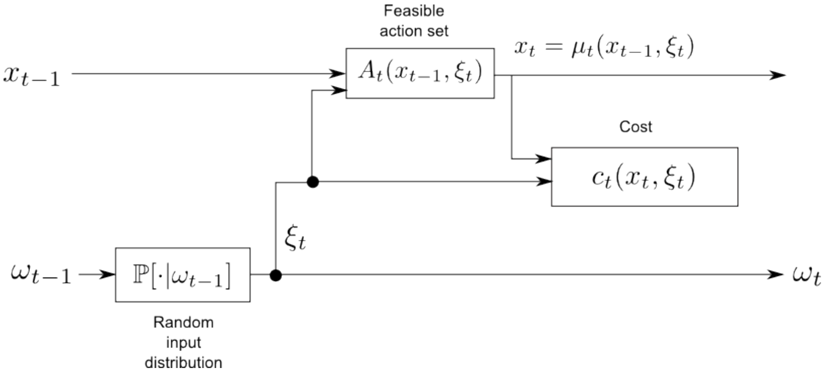
$$T_{t-1}(\omega_t)x_{t-1}(\omega_{[t-1]}) + W_t(\omega_t)x_t(\omega_{[t]}) = h_t(\omega_t), \omega_{[t]} \in \Xi_{[t]}$$

$$\vdots$$

$$T_{H-1}(\omega_H)x_{H-1}(\omega_{[H-1]}) + W_H(\omega_H)x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in \Xi_{[H]}$$

$$x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \ldots, H$$

Multistage Stochastic Linear Programming



Application of Dynamic Programming to Multi-Stage Stochastic Linear Programming

Step 1-a: Compute Q_H

$$\begin{aligned} Q_H(x_{H-1}, \xi_H) = \quad & \min_{x_H} c_H(\omega_H)^T x_H \\ & \text{s.t. } T_{H-1}(\omega_H)x_{H-1} + W_H(\omega_H)x_H = h_H(\omega_H) \\ & x_H \geq 0 \end{aligned}$$

Step 1-b: Compute V_H

$$V_H(x_{H-1}, \omega_{H-1}) = \mathbb{E}_{\xi_H}[Q_H(x_{H-1}, \xi_H)|\omega_{H-1}]$$

Recursive step a: Compute Q_t:

$$\begin{aligned} Q_t(x_{t-1},\xi_t) = \quad & \min_{x_t} c_t(\omega_t)^T x_t + V_{t+1}(x_t,\omega_t) \\ & \text{s.t. } T_{t-1}(\omega_t)x_{t-1} + W_t(\omega_t)x_t = h_t(\omega_t) \\ & x_t \geq 0 \end{aligned}$$

Recursive step b: Compute V_t:

$$V_t(x_{t-1},\omega_{t-1}) = \mathbb{E}_{\xi_t}[Q_t(x_{t-1},\xi_t)|\omega_{t-1}]. \tag{1}$$

Final step: Solve for x_1:

$$\begin{aligned} & \min c_1^T x_1 + V_2(x_1) \\ & \text{s.t. } W_1 x_1 = h_1 \\ & x_1 \geq 0 \end{aligned}$$

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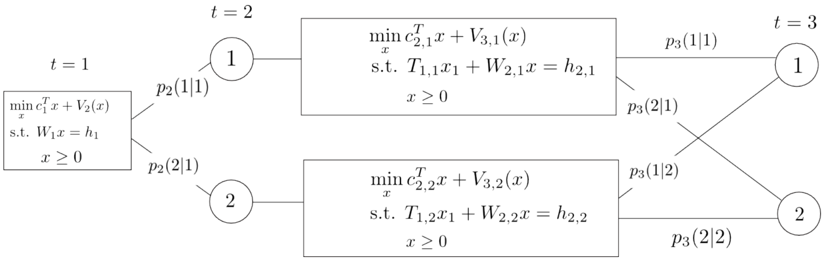
Value Functions on a Lattice



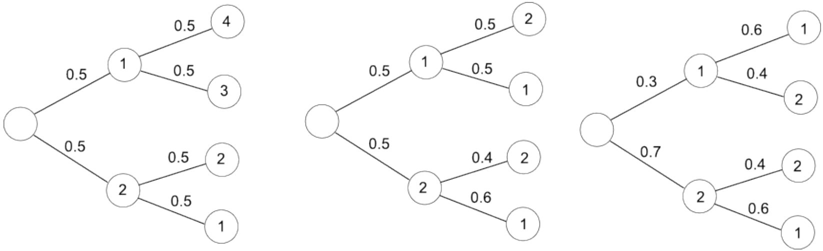
Notation:

- $V_{t+1,k}$: value function of stage $t+1$, given realization k in stage t
- Circles: realizations of uncertainty
- Boxes: decisions
- $p_t(j|i)$: transition probability from node i of stage $t-1$ to node j of stage t

Serial Independence

Serial independence: distribution of ξ_t is independent of the history of realizations $\xi_{[t-1]}$ for all stages t:

$$\mathbb{P}[\xi_t = i | \xi_{[t-1]}] = p_t(i), \forall \xi_{[t-1]} \in \Xi_{[t-1]}, i \in \Xi_t, t = 2, \ldots, H.$$

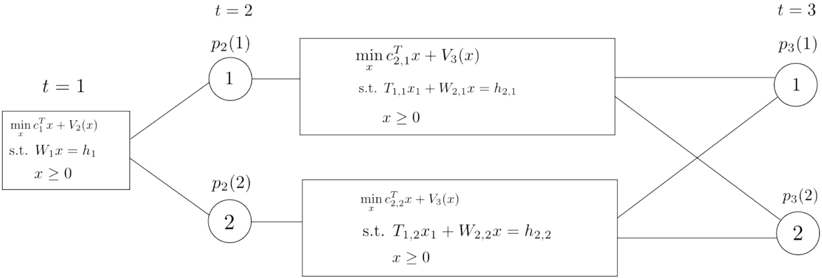


Which trees satisfy serial independence?

Implications of Serial Independence

Serial independence $\Rightarrow$

- No transition probabilities, only node probabilities
- Value functions $V_{t+1}(x_t)$, instead of $V_{t+1,k}(x_t)$



Intuition: future is identical regardless of $\xi_t \Rightarrow$ future cost independent of ξ_t

Structure of Value Function

Consider a multi-stage stochastic linear program defined on a lattice, and denote Ξ_t as the set of possible realizations in stage t. If Ξ_t is finite for all t then

- $V_{t+1,\omega_t}(x_t)$ and $Q_{t+1}(x_t, \xi_{t+1})$ are piecewise linear (pwl) convex
- dom V_{t+1,ω_t} and dom Q_{t+1} are polyhedral

Proof is by induction, excellent activity for Saturday night

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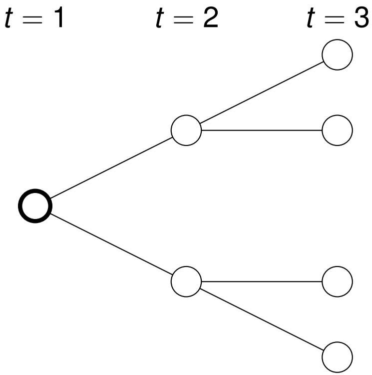
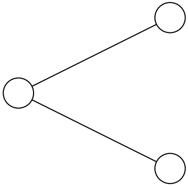
Scenario Tree Model of Multi-Stage Stochastic Program



Goal: know what to do in the root node: $t = 1, k = 1$

Building Block

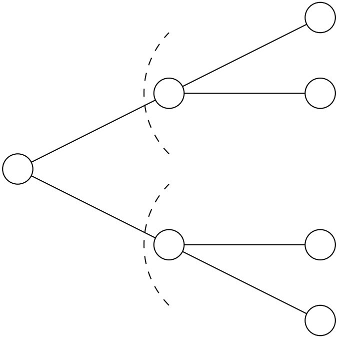
We know how to solve a 2-stage stochastic program



Algorithms

- L-shaped method
- Multi-cut L-shaped method

Breaking Down Multi-Stage to 2-Stage

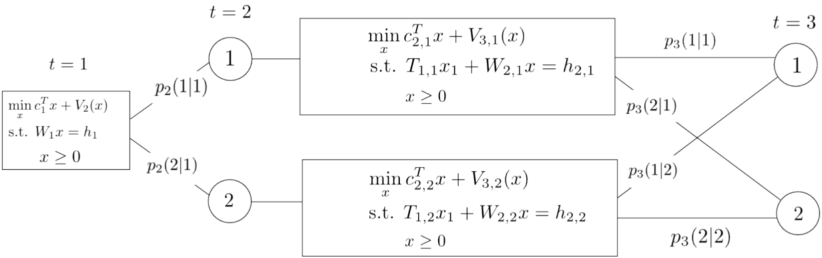


First index denotes time, second index denotes scenario

- Cost-to-go at $t = 2, k = 1$: piecewise linear function of $x_{2,1}$
- Cost-to-go at $t = 2, k = 2$: piecewise linear function of $x_{2,2}$
- Problem at $t = 1, k = 1$ has identical structure to 2-stage stochastic program

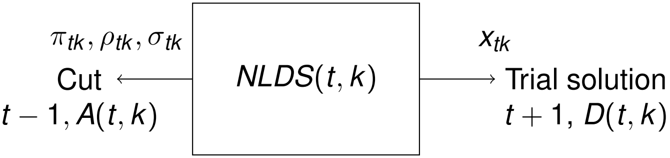
Idea of Nested Decomposition

- Each box corresponds to a linear program (why?)
- Nested decomposition: repeated application of the L-shaped method
- Variants depending on how we traverse the scenario tree



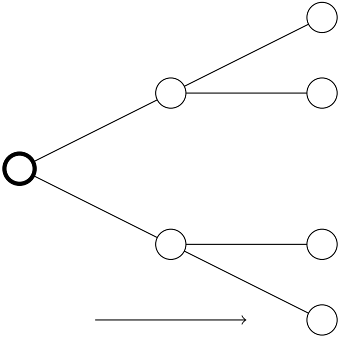
Nested L-Shaped Decomposition Subproblem (NLDS)

Building block: $NLDS(t, k)$: problem at stage t, scenario k



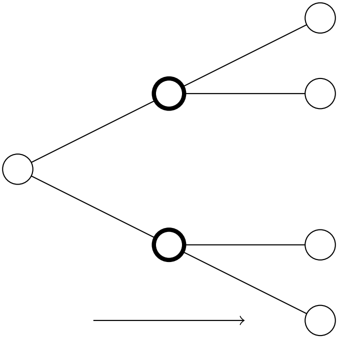
- $A(t, k)$: ancestor of outcome k in period t
- $D(t, k)$: descendants of outcome k in period t

Example



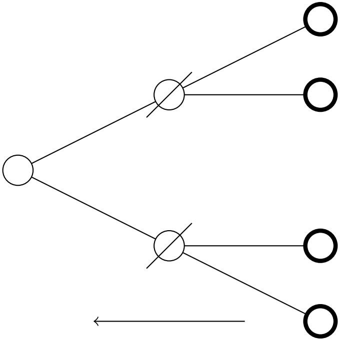
- Node: $(t = 1, k = 1)$
- Direction: forward
- Output: $x_{1,1}$

Example



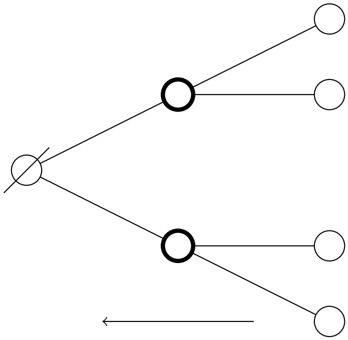
- Nodes: $(t = 2, k)$, $k \in \{1, 2\}$
- Direction: forward
- Output: $x_{2,k}$, $k \in \{1, 2\}$

Example



- Nodes: $(t = 3, k)$, $k \in \{1, 2, 3, 4\}$
- Direction: backward
- Output: $(\pi_{3,k}, \rho_{3,k}, \sigma_{3,k})$, $k \in \{1, 2, 3, 4\}$

Example



- Nodes: $(t = 2, k)$, $k \in \{1, 2\}$
- Direction: backward
- Output: $(\pi_{2,k}, \rho_{2,k}, \sigma_{2,k})$, $k \in \{1, 2\}$

Example: Newsboy Problem

Denote:

- C: unit cost of newspapers
- P: sales price of newspapers
- D_ω: random demand
- x: amount of newspapers procured (first stage)
- s: amount of papers sold (second stage)

Write out *NLDS* for stage 1 and 2

First stage:

$$NLDS(1) : \min_x C \cdot x$$
$$\text{s.t. } x \geq 0$$

Second stage:

$$NLDS(2,k) : \min_s -P \cdot s$$
$$\text{s.t. } s \leq D_k$$
$$s \leq x$$
$$s \geq 0$$

Example: Hydrothermal Scheduling

Denote

- C: marginal cost of thermal units
- E: reservoir capacity of hydroelectric dam
- $R_{t,k}$: rainfall (random)
- D_t: power demand
- x: hydro power stored in the dam
- q: hydro power production
- p: thermal production

Write out *NLDS* for stage t

NLDS for stage t and outcome k:

$$NLDS(t,k): \min_{x,q,p} C \cdot p$$
$$\text{s.t. } x \leq E$$
$$x \leq x_{t-1} + R_{t,k} - q$$
$$p + q \geq D_t$$
$$x, q, p \geq 0$$

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The Nested L-Shaped Decomposition Subproblem

For each stage $t = 1, \ldots, H-1$, scenario $k = 1, \ldots, |\Xi_t|$

$$
\begin{aligned}
NLDS(t,k): \quad & \min_{x,\theta} (c_{t,k})^T x + \theta \\
(\pi): \quad & W_{t,k} x = h_{t,k} - T_{t-1,k} x_{t-1,A(t,k)} \\
(\rho_j): \quad & E_{t,k,j} x + \theta \geq e_{t,k,j}, j = 1, \ldots, r_{t,k} && (2) \\
(\sigma_j): \quad & D_{t,k,j} x \geq d_{t,k,j}, j = 1, \ldots, s_{t,k} && (3) \\
& x \geq 0
\end{aligned}
$$

- Ξ_t: distinct realizations of ξ_t
- $A(t,k)$: ancestor of realization k at stage t
- $x_{t-1,A(t,k)}$: current solution from $A(t,k)$
- Constraints (3): feasibility cuts
- Constraints (2): optimality cuts

Boundary Conditions

- For $t = 1$, $h_{t,k} - T_{t-1,k} x_{t-1,A(t,k)}$ is replaced by b
- For $t = H$, θ and constraints (2) and (3) are removed

Dual of $NLDS(t,k)$

$$\max_{\pi,\rho,\sigma} \pi^T(h_{t,k} - T_{t-1,k}x_{t-1,A(t,k)}) + \sum_{j=1}^{r_{t,k,j}} \rho_j^T e_{t,k} + \sum_{j=1}^{s_{t,k}} \sigma_j^T d_{t,k,j}$$

$$\text{s.t. } \pi^T W_{t,k} + \sum_{j=1}^{r_{t,k}} \rho_j^T E_{t,k,j} + \sum_{j=1}^{s_{t,k}} \sigma_j^T D_{t,k,j} \leq c_{t,k}^T$$

$$\sum_{j=1}^{r_{t,k}} 1^T \rho_j = 1$$

$$\rho_1, \ldots, \rho_{r_{t,k}} \geq 0$$

$$\sigma_1, \ldots, \sigma_{s_{t,k}} \geq 0$$

Feasibility Cuts

If $NLDS(t,k)$ is infeasible, solver returns $(\pi, \sigma_1, \ldots, \sigma_{s_{t,k}})$ with $\sigma_j \geq 0, j = 1, \ldots, s_{t,k}$, such that:

- $\pi^T(h_{t,k} - T_{t-1,k}x_{t-1,A(t,k)}) + \sum_{j=1}^{s_{t,k}} \sigma_j^T d_{t,k,j} > 0$
- $\pi^T W_{t,k} + \sum_{j=1}^{s_{t,k}} \sigma_j^T D_{t,k,j} \leq 0$

The following is a valid feasibility cut for $NLDS(t-1, a(k))$:

$$(FC) : D_{t-1,A(t,k)}x \leq d_{t-1,A(t,k)}$$

where

$$\begin{aligned} D_{t-1,A(t,k)} &= \pi^T T_{t-1,k} \\ d_{t-1,A(t,k)} &= \pi^T h_{tk} + \sum_{j=1}^{s_{t,k}} \sigma_j^T d_{t,k,j} \end{aligned}$$

Optimality Cuts

For all $k \in D_{t-1,j}$, solve $NLDS(t,k)$, then compute

$$E_{t-1,j} = \sum_{k \in D(t-1,j)} p_t(k|j) \cdot (\pi_{t,k})^T T_{t-1,k}$$

$$e_{t-1,j} = \sum_{k \in D(t-1,j)} p_t(k|j) \cdot ((\pi_{t,k})^T h_{t,k} + \sum_{i=1}^{r_{t,k}} \rho_{t,k,i}^T e_{t,k,i} + \sum_{i=1}^{s_{t,k}} \sigma_{t,k,i}^T d_{t,k,i})$$

The following is an optimality cut for $NLDS(t-1,j)$:

$$E_{t-1,j} x + \theta \geq e_{t-1,j}$$

The Nested Decomposition Algorithm

Pass	t	k	Result	Action
F	1		Feasible	$t \leftarrow 2$, $k \leftarrow 1$, Store θ_1, x_1 Send x to $NLDS(2, j), j \in D(1)$
F	1		Infeasible	Infeasible, exit
F	$1 < t \leq H - 1$	$k < \lvert\Xi_t\rvert$	Feasible	$k \leftarrow k + 1$, Send x to $NLDS(t + 1, j), j \in D(t, k)$
F	$1 < t \leq H - 1$	$k < \lvert\Xi_t\rvert$	Infeasible	$k \leftarrow k + 1$ Add FC to $NLDS(t - 1, A(t, k))$
F	$1 < t \leq H - 1$	$\lvert\Xi_t\rvert$	Feasible	$t \leftarrow t + 1$, $k \leftarrow 1$ Send x to $NLDS(t + 1, j), j \in D(t, k)$ If $t = H - 1$ then Pass $\leftarrow$ B
F	$1 < t \leq H - 1$	$\lvert\Xi_t\rvert$	Infeasible	$t \leftarrow t + 1$, $k \leftarrow 1$ Add FC to $NLDS(t - 1, A(t, k))$ If $t = H - 1$ then Pass $\leftarrow$ B
B	$t \geq 2$	$k < \lvert\Xi_t\rvert$	Feasible	$k \leftarrow k + 1$, Store (π, ρ, σ)
B	$t \geq 2$	$k < \lvert\Xi_t\rvert$	Infeasible	$k \leftarrow k + 1$ Add FC to $NLDS(t - 1, A(t, k))$
B	2	$\lvert\Xi_t\rvert$	Feasible	Pass $\leftarrow$ F, $t \leftarrow 1$ Add OC to $NLDS(1)$ If $\theta_1 \geq e - Ex_1$: Optimal, exit
B	2	$\lvert\Xi_t\rvert$	Infeasible	Pass $\leftarrow$ F, $t \leftarrow 1$ Add FC to $NLDS(1)$
B	$t > 2$	$\lvert\Xi_t\rvert$	Feasible	$t \leftarrow t - 1$, $k \leftarrow 1$ Add OC to $NLDS(t - 1, A(t, k))$
B	$t > 2$	$\lvert\Xi_t\rvert$	Infeasible	$t \leftarrow t - 1$, $k \leftarrow 1$ Add FC to $NLDS(t - 1, A(t, k))$

Direction of Movement

Whenever $NLDS(t, k)$ is solved, the following data is generated

- If feasible:
  - Trial decision $x_{t,k}$ (can be sent forward)
  - Optimality cut (can be sent backwards)
- If infeasible: feasibility cut (can be sent backwards)

Alternative protocols

- **Fast-forward-fast-back**: move in current direction, as far as possible
- **Fast-forward**: move forward whenever possible
- **Fast-back**: move backwards whenever possible

If all Ξ_t are finite sets and all x have finite upper bounds, then the nested L-shaped method converges finitely to an optimal solution

Proof: BL, page 268

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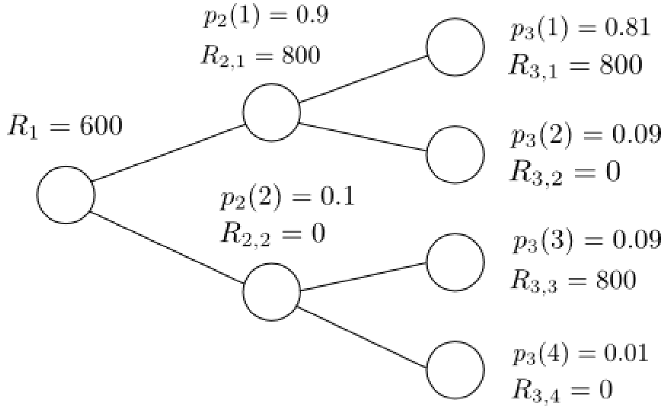
Hydrothermal Scheduling over Three Periods

Consider the following hydrothermal problem:

- Demand: 1000 MW
- Energy capacity of dam: 750 MWh
- Marginal cost of thermal production: 25 $/MWh
- Capacity of thermal units: 500 MW
- Marginal cost of unserved demand: 1000 $/MWh

Scenario Tree



Is the tree serially independent?

NLDS

NLDS for first period:

$$NLDS(1) : \min 25 \cdot p + 1000 \cdot l$$

$$\text{s.t. } x \leq 750$$

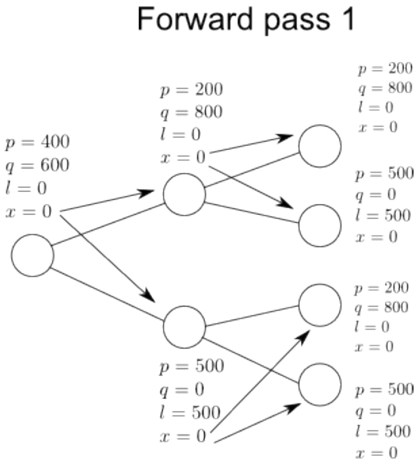
$$x \leq 600 - q$$

$$p + q + l \geq 1000$$

$$p \leq 500$$

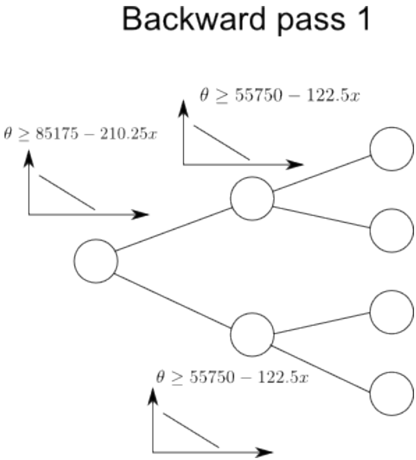
$$x, p, q, l \geq 0$$

Algorithm Progress: Forward Pass 1



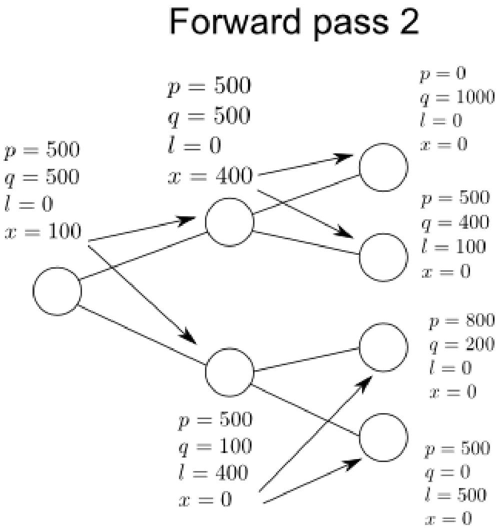
Greedy behavior ⇒ load shedding in stage 2, node 2, and stage 3, nodes 2 and 4

Algorithm Progress: Backward Pass 1



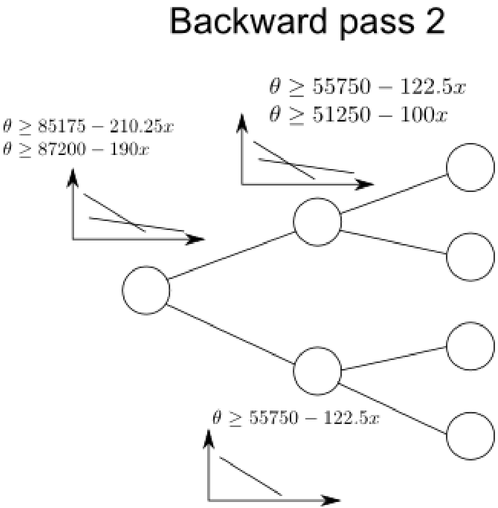
Cuts generated in stage 2 are identical (why?)

Algorithm Progress: Forward Pass 2



Note utilization of hydro in stage 1

Algorithm Progress: Backward Pass 2



New optimality cuts: node 1 of stage 2, stage 1

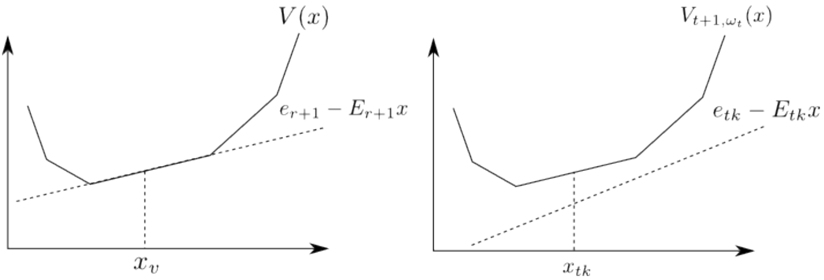
Convergence and Optimal Solution

Third forward pass $\rightarrow$ no new cut $\Rightarrow$ convergence

Load shedding in optimal policy: nodes 2 and 4 of stage 3

Optimal policy prevents spillage in scenarios of abundant water supply (node 1 of stage 3)

Optimality Cuts of L-Shaped Method and Nested Decomposition



- L-shaped method: optimality cuts support value function
- Nested decomposition: optimality cuts may be strictly below the value function