The Multicut L-Shaped Method

Operations Research

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Extensive Form 2-Stage Stochastic Linear Program

$$(EF): \quad \min c^T x + \mathbb{E}_\omega[\min q(\omega)^T y(\omega)]$$
$$Ax = b$$
$$T(\omega)x + W(\omega)y(\omega) = h(\omega)$$
$$x \geq 0, y(\omega) \geq 0$$

- First-stage decisions: $x \in \mathbb{R}^{n_1}$
- second-stage decisions: $y(\omega) \in \mathbb{R}^{n_2}$
- First-stage parameters: $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$, $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage parameters: $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$, $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$ and $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

L-Shaped Master Problem

We know that

$$V(x) = \{\sum_{\omega} p_\omega \min q_\omega^T y_\omega | W_\omega y_\omega = h_\omega - T_\omega x, y_\omega \geq 0\}$$

is a *piecewise linear* function of x

Define **master problem** as

$$\begin{aligned}
(M): \quad & z_k = \min c^T x + \theta \\
& Ax = b \\
& \sigma^T (h - Tx) \leq 0, \sigma \in R_k \subseteq R & (1) \\
& \theta \geq \pi^T (h - Tx), \pi \in V_k \subseteq V & (2) \\
& x \geq 0
\end{aligned}$$

- Feasibility cuts: equation 1
- Optimality cuts: equation 2

Multicut L-Shaped Master Problem

We also know that

$$Q_\omega(x) = \{\min q_\omega^T y | W_\omega y = h_\omega - T_\omega x, y \geq 0\}$$

is a *piecewise linear* function of x

$$
\begin{aligned}
(M): \quad & \min c^T x + \sum_{\omega=1}^{N} p_\omega \theta_\omega \\
& Ax = b \\
& \sigma^T (h_\omega - T_\omega x) \leq 0, \sigma \in R_{\omega k} \subseteq R_\omega \\
& \theta_\omega \geq \pi^T (h_\omega - T_\omega x), \pi \in V_{\omega k} \subseteq V_\omega \\
& x \geq 0
\end{aligned}
$$

L-Shaped Optimality Cuts

Consider a trial first-stage decision x^v

Let π_ω be simplex multipliers of second-stage problem:

$$\min q_\omega^T y$$
$$\text{s.t. } W_\omega y = h_\omega - T_\omega x^v$$
$$y \geq 0$$

Then $\sum_\omega p_\omega \pi_\omega^T (h_\omega - T_\omega x)$ supports $V(x)$ at x^v

Multicut L-Shaped Optimality Cuts

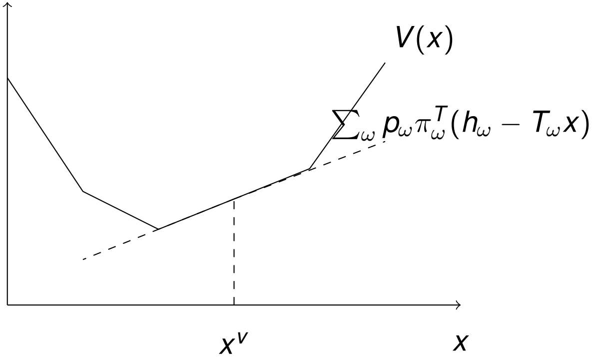
Consider a trial first-stage decision x^v

Let π_ω be simplex multipliers of second-stage problem:

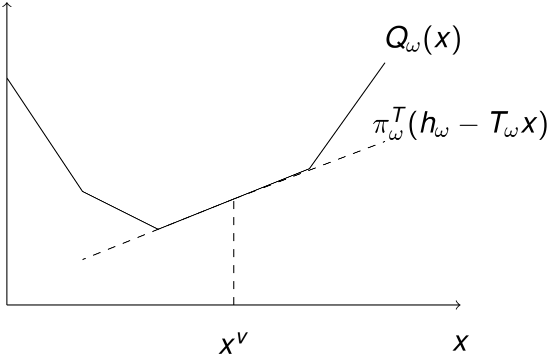
$$
\begin{aligned}
&\min q_\omega^T y \\
&\text{s.t. } Wy = h_\omega - T_\omega x^v \\
&y \geq 0
\end{aligned}
$$

Then $\pi_\omega^T(h_\omega - T_\omega x)$ supports $Q_\omega(x)$ at x^v

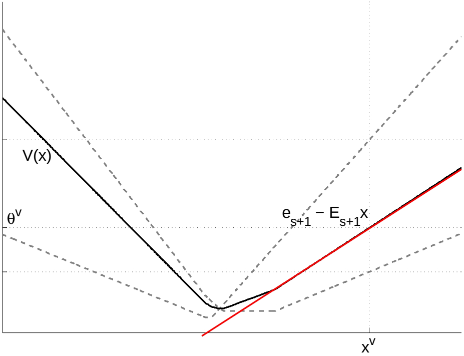
L-Shaped Method: Graphical Illustration of Optimality Cuts



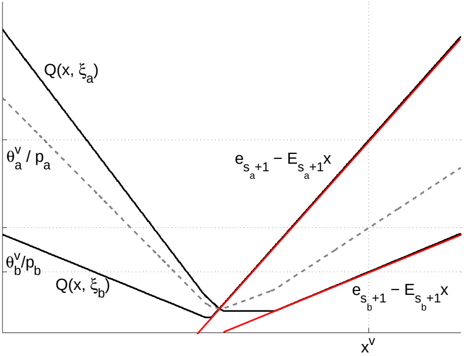
Multicut L-Shaped Method: Graphical Illustration of Optimality Cuts



L-Shaped Versus Multicut



L-Shaped Versus Multicut



The L-Shaped Algorithm

Step 0: Set $k = 0$, $V_0 = R_0 = \emptyset$

Step 1: Solve (M)

- If (M) is feasible, store x_k
- If (M) is infeasible, exit: infeasible

Step 2: For $\omega = 1, \ldots, N$, solve (S_ω) with x_k as input

- If (S_ω) is infeasible, let $S_{k+1} = S_k \cup \{\sigma_{k+1}\}$, where σ_{k+1} is an extreme ray of (S_ω), let $k = k + 1$ and return to step 1
- If (S_ω) is feasible, store $\pi_{\omega,k+1}$

Step 3: Let $V_{k+1} = V_k \cup \{(p_1\pi_{1,k+1}, \ldots, p_N\pi_{N,k+1})\}$

- If $V_k = V_{k+1}$ then terminate with (x_k, y_{k+1}) as the optimal solution.
- Else, let $k = k + 1$ and return to step 1

The Multicut L-Shaped Algorithm

Step 0: Set $k = 0$, $V_{\omega 0} = R_{\omega 0} = \emptyset$ for all ω.

Step 1: Solve (M).

- If (M) is feasible, store x_k.
- If (M) is infeasible, exit. The problem is infeasible.

Step 2: For $\omega = 1, \ldots, N$, solve (S_ω) with x_k as input.

- If (S_ω) is infeasible, let $S_{\omega,k+1} = S_{\omega k} \cup \{\sigma_{\omega,k+1}\}$. Let $k = k + 1$ and return to step 1.
- If (S_ω) is feasible, store $\pi_{\omega,k+1}$.

Step 3: For $\omega = 1, \ldots, N$, let $V_{\omega,k+1} = V_{\omega k} \cup \{\pi_{\omega,k+1}\}$.

- If $V_{\omega k} = V_{\omega,k+1}$ for all ω then terminate with (x_k, y_{k+1}) as the optimal solution.
- Else, let $k = k + 1$ and return to step 1.

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Example: Birge-Louveaux

$$
\begin{aligned}
z &= \min \mathbb{E}_\xi (y_1 + y_2) \\
\text{s.t. } &0 \le x \le 10 \\
&y_1 - y_2 = \xi - x \\
&y_1, y_2 \ge 0
\end{aligned}
$$

$$
\xi = \begin{cases} 1 & p_1 = 1/3 \\ 2 & p_2 = 1/3 \\ 4 & p_3 = 1/3 \end{cases}
$$

$K_2 = \mathbb{R}$

Multicut L-Shaped Method in Example 2

- Iteration 1, Step 1: $x^1 = 0$
- Iteration 1, Step 3: x^1 not optimal, add cuts:

$$\theta_1 \geq \frac{1-x}{3}, \theta_2 \geq \frac{2-x}{3}, \theta_3 \geq \frac{4-x}{3}$$

- Iteration 2, Step 1: $x^2 = 10$, $\theta_1^2 = -3$, $\theta_2^2 = -8/3$, $\theta_3^2 = -2$
- Iteration 2, Step 3: x^2 not optimal, add cuts:

$$\theta_1 \geq \frac{x-1}{3}, \theta_2 \geq \frac{x-2}{3}, \theta_3 \geq \frac{x-4}{3}$$

- Iteration 3, Step 1: $x^3 = 2$, $\theta_1^3 = 1/3$, $\theta_2^3 = 0$, $\theta_3^3 = 2/3$ is optimal

Tradeoffs

Multicut L-shaped method has:

- More detailed representation of value function (+)
- Larger master problem (-)

Typically (not always), fewer iterations are required in multicut L-shaped method, but each iteration requires more time

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Master Problem

$$(M): \quad \min_{x \geq 0} \sum_{i=1}^{n} I_i \cdot x_i + \sum_{\omega=1}^{N} p_\omega \theta_\omega$$

$$\theta_\omega \geq \sum_{j=1}^{m} \lambda_{\omega j}^{v} D_j + \sum_{i=1}^{n} \rho_{\omega i}^{v} x_i, v \in V_{\omega k}$$

$$\theta_\omega \geq 0$$

Sequence of Investment Decisions

Iteration	Coal (MW)	Gas (MW)	Nuclear (MW)	Oil (MW)
1	0	0	0	0
2	0	0	0	10701.3
3	0	14309.6	0	0
4	10407	0	0	0
5	0	0	7154.8	5034.6
6	0	2329	7154.8	1410
7	0	1280.2	8518.8	1647.6
8	2102.1	3310.9	5756	0
9	8767.3	236.7	0	2291.4
10	6396	0	3919	1168.8
11	8230.5	2165	773.5	0
12	5085	1311	3919	854

Sequence of Value Function Approximations

Iteration	L-shaped θ_k	Multicut $\sum_{\omega=1}^{N} p_\omega \theta_{\omega k}$
1	0	0
2	0	0
3	0	14674
4	0	61181
5	0	61181
6	0	28444
7	59736	83545
8	40998	186865
9	50222	108401
10	96290	171767
11	61593	125272
12	186788	
13	107349	
14	124788	
15	130041	
16	125272	

Observations

- Multi-cut converges with fewer iterations
- Multi-cut incurs non-zero second stage cost in iteration 3 (L-shaped method requires 7 iterations)
- Iterations 5 and 6 have identical $\sum_{\omega=1}^{N} p_\omega \theta_{\omega k}$, does not imply convergence
- θ_k for L-shaped need not be increasing (see iteration 12, attempt to remove nuclear)
- Final iterations of L-shaped (12-15) oscillate around near-optimal mix