

Linear Programming

Operations Research

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1 Primal Linear Program

2 Dual Linear Program

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Linear Programs in Standard Form

Linear program (LP) in standard form:

$$\begin{aligned}(P) : \quad & \min z = c^T x \\ & \text{s.t. } Ax = b \\ & x \geq 0\end{aligned}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Any LP can be expressed in standard form

Solution: a vector x such that $Ax = b$

Feasible solution: a solution with $x \geq 0$

Optimal solution: a feasible solution x^* such that $c^T x^* \leq c^T x$
for all feasible solutions x

Basis and Basic Solution

Basis: a choice of n linearly independent columns of A

Denote $A = [B, N]$ where N are non-basic columns

Each basis corresponds to a **basic solution** $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ with

$$x_B = B^{-1}b \text{ and } x_N = 0$$

Geometric property: Basic feasible solutions correspond to extreme points of the feasible region $\{x | Ax = b, x \geq 0\}$

A basis is

- **feasible** if $B^{-1}b \geq 0$
- **optimal** if feasible and $c_N^T - c_B^T B^{-1}N \geq 0$

Optimal Basis

Claim: $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ is optimal if $B^{-1}b$ and $c_N^T - c_B^T B^{-1}N \geq 0$

Proof: $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$ is obviously feasible, can we improve objective function by moving away from it?

Idea: substitute basic variables for non-basic variables in objective function $c^T x$:

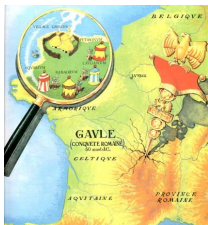
$$\begin{aligned} [B \quad N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} &= b \Leftrightarrow \\ Bx_B + Nx_N &= b \Leftrightarrow \\ x_B &= B^{-1}(b - Nx_N) \end{aligned} \quad (1)$$

Substituting equation (1) into the objective function,

$$\begin{aligned}c^T x &= c_B^T x_B + c_N^T x_N \\ &= c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N\end{aligned}\quad (2)$$

- Non-basic variables can only increase when moving away from the current solution while remaining feasible
- Since $c_N^T - c_B^T B^{-1} N \geq 0$, second term of equation (2) can only increase when moving in the neighborhood of the current solution \Rightarrow current solution is locally optimal
- In linear programming, locally optimal \Rightarrow globally optimal

Our Focus: $z(b)$



We care about how the optimal value of a linear program depends on the right-hand side parameters b :

$$\begin{aligned} z(b) &= \min c^T x \\ &\text{s.t. } Ax = b \\ &x \geq 0 \end{aligned}$$

Example: The Diet Problem

Problem: Choose 3 dishes (x_1, x_2, x_3) so as to satisfy nutrient requirements b_1 and b_2 , while minimizing cost (dishes cost 1 \$, 2 \$, and 3 \$ respectively)

Table: The unit of nutrients in each dish.

| | Dish 1 | Dish 2 | Dish 3 |
|------------|--------|--------|--------|
| Nutrient 1 | 0.5 | 4 | 1 |
| Nutrient 2 | 2 | 1 | 2 |

$$\min x_1 + 2x_2 + 3x_3$$

$$\text{s.t. } 0.5x_1 + 4x_2 + x_3 = b_1$$

$$2x_1 + x_2 + 2x_3 = b_2$$

$$x_1, x_2 \geq 0$$

Example: The Diet Problem - Basic Solutions

Three possible bases:

$$B_1 = \begin{bmatrix} 0.5 & 4 \\ 2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 & 1 \\ 2 & 2 \end{bmatrix}, B_3 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Three candidate basic solutions, (parametrized on (b_1, b_2)):

$$x_{B_1} = \begin{bmatrix} -0.1333b_1 + 0.5333b_2 \\ 0.2667b_1 - 0.0667b_2 \end{bmatrix}$$
$$x_{B_2} = \begin{bmatrix} -2b_1 + b_2 \\ 2b_1 - 0.5b_2 \end{bmatrix}$$
$$x_{B_3} = \begin{bmatrix} 0.2857b_1 - 0.1429b_2 \\ -0.1429b_1 + 0.5714b_2 \end{bmatrix}$$

We want to understand how the objective function behaves as we change b

Example: The Diet Problem - Basic Feasible Solutions

In order for a basic solution to be feasible, it is necessary that b is such that $x_B \geq 0$

Denote $R_i = \{(b_1, b_2) : x_{B_i} \geq 0\}$, then:

$$R_1 = \{0.25b_1 \leq b_2 \leq 4b_1\}$$

$$R_2 = \{2b_1 \leq b_2 \leq 4b_1\}$$

$$R_3 = \{0.25b_1 \leq b_2 \leq 2b_1\}$$

Example: The Diet Problem - Basic Optimal Solutions

Cost of each basic solution, parametric on (b_1, b_2) :

$$c_{B_1}^T x_{B_1} = 0.4b_1 + 0.4b_2$$

$$c_{B_2}^T x_{B_2} = 4b_1 - 0.5b_2$$

$$c_{B_3}^T x_{B_3} = 0.1429b_1 + 1.4286b_2$$

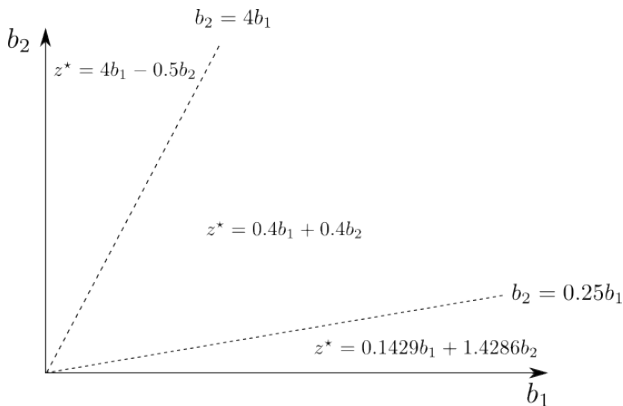
Denote S_i as region for which B_i is optimal:

$$S_1 = \{0.25b_1 \leq b_2 \leq 4b_1\}$$

$$S_2 = \{b_2 \leq 4b_1\}$$

$$S_3 = \{b_2 \leq 0.25b_1\}$$

Example: The Diet Problem - $z(b)$



$z(b)$ is piecewise linear convex function of b

Pop quiz: for any $b \notin S_1$ the basic solutions are not feasible, what can you conclude about dish 3?

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The **dual** of problem (P) is the following linear program:

$$(D) : \max \pi^T b$$

$$\text{s.t. } \pi^T A \leq c^T$$

If primal problem is not in standard form, use the following rules

| Primal | Minimize | Maximize | Dual |
|-------------|------------|------------|-------------|
| Constraints | $\geq b_j$ | ≥ 0 | Variables |
| | $\leq b_j$ | ≤ 0 | |
| | $= b_j$ | Free | |
| Variables | ≥ 0 | $\leq c_j$ | Constraints |
| | ≤ 0 | $\geq c_j$ | |
| | Free | $= c_j$ | |

Example: Dual of the Diet Problem

Recall the diet problem:

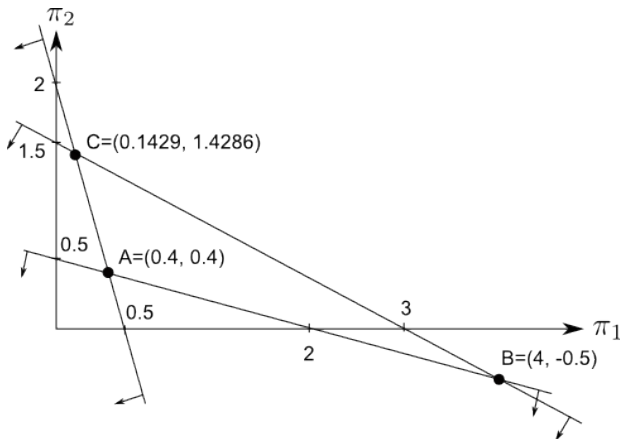
$$\begin{aligned} \min \quad & x_1 + 2x_2 + 3x_3 \\ \text{s.t.} \quad & 0.5x_1 + 4x_2 + x_3 = b_1 \\ & 2x_1 + x_2 + 2x_3 = b_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Using the table in the previous slide, the dual is:

$$\begin{aligned} \max \quad & b_1\pi_1 + b_2\pi_2 \\ \text{s.t.} \quad & 0.5\pi_1 + 2\pi_2 \leq 1 \\ & 4\pi_1 + \pi_2 \leq 2 \\ & \pi_1 + 2\pi_2 \leq 3 \end{aligned}$$

Example: Dual of the Diet Problem - Feasible Region

Figure: The dual feasible region of the diet problem. Each point A , B and C is a basic solution of the dual feasible region and corresponds to a basis of the primal problem in standard form.



Rewriting the Dual Problem

Claim: $z(b)$ is a piecewise linear convex function

Proof: If a dual optimal solution exists, then one dual basic solution¹ must be optimal

Reformulation of the dual problem:

$$\max_{i=1, \dots, r} \pi_i^T b$$

where r indexes *finitely many* basic feasible solutions

¹General definition of **basic solution** for a polyhedron $P \subset \mathbb{R}^n$ (not necessarily in standard form) that is defined by linear equalities and inequalities: a vector x such that (i) all equality constraints are active and (ii) out of the constraints that are active at x , n are linearly independent

Computing Dual Basic Solutions

For linear programs in standard form, each basis B of the primal coefficient matrix A corresponds to a basic solution of the dual feasible set, according to the relationship

$$\pi^T = c_B^T B^{-1}$$

and vice versa

Example: Dual of the Diet Problem - Basic Solutions

Recall the three possible bases of the diet problem:

$$B_1 = \begin{bmatrix} 0.5 & 4 \\ 2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 & 1 \\ 2 & 2 \end{bmatrix}, B_3 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Basic solutions of the dual feasible region can be computed according to $\pi = c_B^T B^{-1}$:

$$A = (0.4, 0.4), B = (4, -0.5), C = (0.1429, 1.4286)$$